

ON THE GEOMETRY OF  
THREE-DIMENSIONAL BOL ALGEBRAS  
WITH SOLVABLE ENVELOPING LIE  
ALGEBRAS OF SMALL DIMENSION

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# 1 INTRODUCTION

The notion of quasigroup and the tangential structure appearing to it have been intensively developing for the 40 past years. The quasigroup as a nonassociative algebraical structure is naturally the generalization of the notion of group. It first appeared in the work of R. Moufang (1935) [88]. She obtained some identities. The smooth local loops were first introduced in the work of Malcev A.I. [27], in connection with the generalization of Lie group.

Left binary algebras were later on called Malcev algebras. It is well known that modern differential geometry and nonassociative algebras are interacting on one another. The notion of binary-ternary operation tangent to the given geodesic loop connected with an arbitrary point in the affine connected space was introduced in Akinis works see [1, 2]. Several mathematicians worked in the development of differential geometry and the study of different classes of quasigroups and loops [26, 49, 54, 59, 60, 63, 68, 82].

In 1925 E. Cartan in his research lay the beginning of investigation of symmetric spaces. Today the given structure plays an important role, in differential geometry and its application. The question arising here is that of, the description and the classification of symmetric space naturally leading to the classification of the corresponding algebraical structure.

The survey on geometry of fiber space stimulates the interest for special types of 3-webs, in particular Bol 3-Webs and the tangential structures appearing to it: Bol algebras [30, 34]. In connection with it the idea of the description of collections of three-dimensional Bol algebras elaborates. For separate classes of Bol algebras see [7, 55]. In this investigation our approach on Bol algebras will be base on the classification of solvable Lie triple systems [10]. As we know Bol algebra can be seen as a Lie triple system equipped with an additional bilinear anti-commutative operation which verify a series of supplementary conditions.

Bol algebras appear under the infinitesimal description of the class of local smooth Bol loop, in the work of L.V. Sabinin and P.O. Mikheev. The interest of studying smooth Bol loop is connected to the fact that, the geodesical loop see(Sabinin [45]) of local symmetric affine connected space, verifies the left Bol identity and automorphic inverse identity. Its an exact algebraic analog of construction of symmetric spaces. In particular, the velocity space in the theory of relativity (STR), is a Bol loop relatively to the additional law of velocity [50]. Thus, relying on what is stated above we are ready to formulate the purpose of this work: Classification of Bol algebras of dimension 3 with solvable Lie algebras of dimension  $\leq 5$ , with accuracy to isomorphism and with accuracy to isotopic which include:

1. the description of three-dimensional Lie triple systems and their corresponding Lie algebras with invomorphisme.

2. The description of three-dimensional Bol algebras linked with the distinguished Lie triple systems above.
3. The construction of the object describing the isotopy of Bol algebras.
4. The description of Bol 3-Webs, connected with the selected Bol algebras.

The methods used in these investigations are the methods of geometry of fiber bound and that of non associative algebras. This work bears a theoretic character. The obtained results can find their use in differential geometry, theory of quasigroup and loop, same as in various applications in physics and mechanics.

The obtained results are new and published in the work of Bouetou Bouetou T. [10, 11, 12, 13, 14, 15, 16, 17, 67], and also in Bouetou Bouetou T. Mikheev P.O. [66].

This thesis constitutes: an introduction, three chapters and their references. Each chapter is divided into paragraphs, and the paragraph are divided into subsections. The enumeration of formulas is connected with the paragraph.

Chapter I constitutes of seven paragraphs.

In §1. **Bol loops** a short discussion about the notion and terminology in quasigroup theory and Bol loop is given.

In §2. **Isotopy of loops** here the definition of isotopy is given. In short form, and proposition, it states the fundament of isotopy of loops.

In §3. **Local analytic Bol loops** here the notion of local analytic Bol loops is discussed. In a definitive manner the notion of Bol algebras as W-algebras [2] verifying a system of identities is introduced. Then the discussion of the imbedding of local analytic Bol loop into a local Lie group and the imbedding of Bol algebra into Lie algebra is dealt with. It shows how to calculate the operations  $\xi \cdot \eta$  and  $(\xi, \eta, \chi)$  of Bol algebra  $\mathfrak{B}$  in the term of enveloping Lie algebra  $\mathfrak{G}$ , his subalgebra  $\mathfrak{h}$  and subspace  $\mathfrak{B}$ .

In §. 4. **3-Webs and coordinates loop of 3-Webs** we see through definitions and short propositions the fundamentals of 3-Webs theory and the coordinates loop of 3-Webs being stated.

In §5. **Isotopy of Bol algebras**. In this paragraph a correct generalization of the notion of isotopy of (global loops to the case of local analytic Bol loops is given. In connection with this, the definition of isotopy of Bol algebras is given. The following theorem is stated and proven:

Let  $B(x)$  and  $\widetilde{B(0)}$  be global analytic Bol loops, and let their tangent Bol algebras be isotopic, then  $\widetilde{B(0)}$  is locally isomorphic, to an analytic Bol loop analytically isotopic to  $B(x)$ .

In §6. **About the classification of Bol algebras**. In this paragraph we state in a short way the method used in this investigation.

In §.7. **Isocline Bol algebras.** This given class of Bol algebras is a particular case of Bol algebras. Any Bol algebra is said to be isocline if and only if it verifies the plane axiom.

Chapter II. This chapter is devoted to the classification of solvable Lie triple systems of dimension 3. It consists of 4 paragraphs.

§.1. **Some information about lie triple systems.** Following the work [26] and [52] the direct and inverse construction of imbedding of Lie triple system into a Lie algebra is presented.

§.2. **Solvable and simple Lie triple systems** On the basis of the analogue theorem of Levi-Malcev theorem [21], which states that if  $\mathfrak{M}$  is a Lie triple system and  $\mathfrak{G} = \mathfrak{M} \dot{+} \mathfrak{h}$  his canonical enveloping Lie algebra and  $r$ -the radical of Lie algebra  $\mathfrak{G}$ , then in  $\mathfrak{G}$  there exist a semisimple subalgebra  $p$ , complementary to  $r$ , such that;

$$\mathfrak{M} = \mathfrak{M}' \dot{+} \mathfrak{M}''$$

where  $\mathfrak{M}' = \mathfrak{M} \cap r$ ,  $\mathfrak{M}'' = \mathfrak{M} \cap p$  and  $\mathfrak{h} = \mathfrak{h}' \dot{+} \mathfrak{h}''$ ,  $\mathfrak{h}' = \mathfrak{M}' \cap r$ ,  $\mathfrak{h}'' = \mathfrak{M}'' \cap p$ .

One can select 3 cases of 3-dimensional Lie triple systems:

1. Semi-simple.
2. Splitting.
3. Solvable.

§.3. **Classification of solvable Lie triple systems.** In this paragraph the following theorem is stated and proven: with accuracy to isomorphism there exists only 7 different types of Lie triple systems of dimension 3.

§.4. **Some example of Bol algebras with solvable trilinear operation.** Here two examples are given and one remark. The first example is of Bol algebras, obtained from the classification of 3-dimensional Lie algebra (Bianchi classification [19]). The second is of Bol algebras, obtained from right alternative algebras [29]. The remark is about the loops obtained from the work [55].

Chapter III. This chapter consists of 7 paragraphs. Each paragraph is based on the result obtained after the proof of theorem II.2.

§.1. **Bol algebras with trivial trilinear operation of type I.** Here it is proven that, with accuracy to isomorphism there exists 6 Bol algebras of type I, and their corresponding 3-webs are described.

§.2. **Bol algebras with trivial trilinear operation of type II.** Here Bol algebras of type with enveloping Lie algebras of dimension 4 and 5 are considered. It's shown that with accuracy to isomorphism there exists 4 Bol algebras with 4 dimensional enveloping Lie algebra, two of them are isotopics. Also, with accuracy to isomorphism there exists 3 families of

Bol algebras with 5 dimensional canonical Lie algebra. Their corresponding 3-Webs are described.

§.3. **Bol algebras with trilinear operation of type III.** For a better investigation of such algebras the case has been divided into type  $III^-$  and type  $III^+$ . In the limit of each of them, Bol algebras with 4 dimensional enveloping Lie algebras are considered. It's shown that in type  $III^+$  and type  $III^-$  with accuracy to isomorphism there exists two families and one exceptional Bol algebra. And the identification of Bol algebras obtained in [29] is made. The corresponding 3-Webs are described.

§.4. **Bol algebras with trilinear operation of type IV.** this examination is divided into two cases type  $IV^+$  and type  $IV^-$ . in the limit of each of them, the examination of Bol algebras with a 4-dimensional canonical Lie algebra is given; therefore it's shown that there exist with accuracy to isomorphism 2 families of Bol algebras. Their corresponding 3-Webs are described.

§.5. **Bol algebras with trilinear operation of type V.** Here also the consideration is divided into type  $V^+$  and type  $V^-$ . In the limit of each of them the examination of Bol algebras with a 4- dimensional canonical Lie algebras is given. And, the corresponding 3-webs are output.

§.6. **Bol algebras with trilinear operation of type VI.** The canonical enveloping Lie algebra of this type of Bol algebras is of dimension 5, therefore it's shown that with accuracy to isomorphism there exist 3 families of Bol algebras.

§.7. **Bol algebras with trilinear operation of type VII.** The canonical enveloping Lie algebra of this type of Bol algebras is of dimension 5, therefore it's shown that with accuracy to isomorphism there exist 6 families and 4 exceptional Bol algebras. And the 3-Webs corresponding to each one of them are output.

## 2 CHAPTER I

### 2.1 BOL LOOP

**Definition I.1.1.** A set  $Q$  with a fixed element  $e$  and binary operations  $(\cdot)$  and  $(\backslash)$ , satisfying the following conditions

$$\forall a \in Q \ e \cdot a = a \cdot e = a \quad (1)$$

$$\forall a, b \in Q \ a \cdot (a \backslash b) = b, a \backslash (a \cdot b) = b \quad (2)$$

is called a left loop with two-sided identity element  $e$ .

**Definition I.1.2.** A left loop with two-sided identity element  $(Q, \cdot, \backslash, e)$  is called a Bol loop if the left Bol identity

$$a \cdot (b \cdot (a \cdot c)) = (a \cdot (b \cdot a)) \cdot c \quad (3)$$

holds for all  $a, b, c \in Q$ .

The following properties are known from the theory of Quasigroups and loops.

1.  $\forall a \in Q$  there is the unique element  $a^{-1} \in Q$  such that:

$$a^{-1} \cdot a = a \cdot a^{-1} = e$$

2.  $\forall a, b \in Q$  the solution of equation  $a \cdot x = b$  is:

$$x = a \backslash b = a^{-1}b,$$

the solution of the equation  $x \cdot a = b$  also exists is uniquely determined and is of the form  $x = a^{-1} \cdot ((a \cdot b) \cdot a^{-1})$ . In particular in any Bol loop one can define the operation of right division by  $a \backslash b = a^{-1} \cdot ((a \cdot b) \cdot a^{-1})$ , then  $(a \cdot b) \backslash b = a$  and  $(a \backslash b) \cdot b = a$

3.  $\forall a \in Q, \forall m \in \mathbb{N}$  defines  $a^m$  recursively by  $a^0 = e, a^m = a^{m-1} \cdot a$  and  $a^{-m} = (a^{-1})^m$ , the left mono-alternative property:

$$a^m \cdot (a^r \cdot b) = a^{m+r} \cdot b \quad (4)$$

holds  $\forall a, b \in Q$  and  $\forall m, r \in \mathbb{Z}$ , in particular, Bol loops are power associative (mono-associative)

$$\forall a \in Q \text{ and } \forall m, r \in \mathbb{Z}, a^m \cdot a^r = a^{m+r} \quad (5)$$

4. Besides the left Bol identity (3) one can consider the right Bol identity

$$((c \cdot a) \cdot b) \cdot a = c \cdot ((a \cdot b) \cdot a) \quad (6)$$

A loop  $(M, \cdot, \backslash, /, e)$  satisfying the properties (3) and (6) simultaneously, is called a Moufang loop. Moufang loops are diassociative, that means, any two elements of a Moufang loop generate a subgroup. Note that diassociativity implies the following properties:

$$\forall a, b \in M \quad (a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$$

$$a \cdot (b \cdot a) = (a \cdot b) \cdot a \quad (\text{elasticity}) \quad (7)$$

5. The Moufang condition (3)+(6) is equivalent to each of the identities:

$$\begin{aligned} a \cdot (b \cdot (a \cdot c)) &= ((a \cdot b) \cdot a) \cdot c, \\ a \cdot (b \cdot (c \cdot b)) &= ((a \cdot b) \cdot c) \cdot b \quad (8) \end{aligned}$$

The Bol condition (3) was first clearly distinguished from stronger Moufang condition (8) in the work [63]. We may say without exaggeration that the Bol loop construction, its various particular cases, modification and generalizations (Moufang loops, M-loops, Bruck loops, semi-Bol loop, etc ...) are the heart of the modern theory of quasigroup and loops. Partly it is explained by isotopic invariance of the Bol property (3) that is, each loop isotopic to a Bol loop is a Bol loop [8, 68] the same result is true for Moufang loops. It is known that the category of left loop is equivalent to the category of equipped homogeneous spaces [42, 43]. This fact is probably the original cause of the theoretical physicist arising interest in the flexible and economic construction of loops of transformations. It seems to us that the loops, in particular Bol loops, will be a tool of the newest natural sciences. (An amazing fact should be pointed out that the symmetrical space is practically some Bol loop!).

## 2.2 IMBEDDING OF LOOPS IN GROUPS

Let  $(Q, \cdot, \backslash, /, e)$  be a left loop and  $L_Q$ -group, generated with the set of left translations

$$L_x : L_x y = x \cdot y, \forall x, y \in Q, L_{x^{-1}} y = x \backslash y$$

The group  $as_l(Q)$ , generated with the set  $l(x, y) \forall x, y \in Q$  where

$$l_{(x, y)} = L_{(x \cdot y)}^{-1} \cdot L_x \cdot L_y$$

is called associant. In fact, it will be called the left associant of the left quasigroup.  $as_l(Q)$  is the subgroup of the group of permutation  $\sigma_Q$ . For  $h \in \sigma_Q, q \in Q$  we will denote the action of  $\sigma_Q$  to  $Q$  by  $hq$ . It's clear that if  $(Q, \cdot, \backslash)$  is associative, then  $as_l(Q) = \{Id_Q\}$  One can verify that



$asl(Q) \subset L_Q \subset \sigma_Q$ . In the work [43] the following application is defined  $m : Q \times \sigma_Q \longrightarrow \sigma_Q$

$$m_q(h) = m(q, h) = L_{hq}^{-1} \cdot h \cdot L_q \cdot h^{-1}, \forall q \in Q, \forall h \in \sigma_Q$$

it's clear that  $m_Q(h) = id_Q$  if and only if  $h$  -is an authomorphis of loop  $Q(\cdot)$ . If  $m_Q(as_l(Q)) = \{id_Q\}$ , then  $Q$  -is a special loop. A group  $H \subset \sigma_Q$  is called left transassociant if a left quasigroup  $Q(\cdot)$ , is such that  $as_l(Q) \subset H$  and  $m_Q(H) \subset H$  and left transassociant of loop  $Q(\cdot)$ , if  $H$  conserves the right neutral element  $e$  of a left loop  $Q(\cdot) : he = e, \forall h \in H, As_l(Q)$  - the minimal left transassociant of a left loop, obtained from  $as_l(Q)$  by an indefinite extension with the help of the operator  $m_Q$  and the operation generating the subgroup from the subset. If in the left loop  $e$  is a right neutral element, then  $as_l(Q)e = e, he = e$  and consequently  $m_Q(h)e = e$  and,  $As_l(Q) \subset L_Q$ . One can prove that for a left loop  $Q(\cdot), m_q(as_l(Q)) \subset as_l(Q), m_q(L_Q) \subset L_Q$  and, hence,  $as_l(Q) = As_l(Q)$  and  $L_Q$  transassociant for  $Q(\cdot)$ .

**Definition 1.1.3.** [Sabinin L.V. [43]] Let  $Q(\times)$  be a left loop, and  $H$  his transassociant, the set  $Q \times H$ , equipped with the internal composition law

$$(q_1, h_1)(q_2, h_2) = (q_1(h_1q_2), \phi(q_1, h_1, q_2, h_2))$$

where

$$\phi(q_1, h_1, q_2, h_2) = l_{q_1 \cdot h_1, q_2} \circ m_{q_2} h_1 \circ h_1 \circ h_2$$

is called a semi-direct product of  $Q$  and  $H$  he is not by  $Q \boxtimes H$ .

**Proposition 1.1.1.** [Sabinin L.V., [43]] The semi-direct product of  $Q \boxtimes H$  of left loop  $Q$  and his transassociant is a group.

About the general properties of Bol loops see [7,66,84,90]

## 2.3 ISOTOPIC LOOPS

Loops  $(Q, \cdot, e)$  and  $(Q', \circ, e)$  are called isotopic if the maps  $U, V$  and  $W : Q \longrightarrow Q'$  exist and bijective such that

$$xU \circ yV = (x \cdot y)W \quad \forall x, y \in Q$$

the triplet  $(U, V, W)$  is called isotopy of  $Q$  and  $Q'$ . If the loops  $(Q, \cdot, e)$  and  $(Q', \circ)$  coincide and  $W = Id_Q$  then the isotopy  $(U, V, Id_Q)$  is called principal.

The following proposition holds.

**Proposition 1.2.1** [Bruck [68]] If the  $(Q', \circ, e)$  is isotopic to the loop  $(Q, \cdot, e)$  then it's isomorphic to the principal isotopy  $(Q, \perp, b.a)$  where the operation  $\perp$  is define by:

$$x \perp y = (x/a) \cdot (b \setminus y)$$

Let  $(Q, \cdot, e)$ -be an abstract loop with the left Bol identity. it's known see [Robinson D.A.[92]] that each loop isotopic to  $Q$  will also verify the left Bol identity. In addition we have the following proposition:

**Proposition I.2.2** [Bruck [68]] If the  $(Q', \perp f)$  is isotopic to the loop  $(Q, \cdot, e)$  with the left Bol identity then it's isomorphic to the principal isotopy  $(Q, \perp, f^2)$  where the operation  $\perp$  is define by:

$$x \perp y = (x/f) \cdot (f \setminus y)$$

## 2.4 Local analytic Bol loops

Let  $Q$  be a set where is defined the operation  $\times$ (multiplication), the left division  $\setminus$  and the unity element  $e$ .

We will say that  $(Q, \times, \setminus, e)$  is a left local loop with a double sided unity if  $Q$  is a topological space with a fixed element  $e$  and for a certain neighborhood  $\mathcal{U}$  of the element  $e$  are defined continuous map

$$\mathcal{U} \times \mathcal{U} \longrightarrow Q : (x, y) \longrightarrow x \times y$$

and

$$\mathcal{U} \times \mathcal{U} \longrightarrow Q : (x, y) \longrightarrow x \setminus y$$

verifying the condition:

1. if  $x \in \mathcal{U}$ , then  $e \times x = x \times e = x$
2. if  $x, y, x \times y \in \mathcal{U}$  then  $x \setminus (x \times y) = y$
3. if  $x, y, x \setminus y \in \mathcal{U}$  then  $x \times (x \setminus y) = y$

If  $Q$  is a smooth manifold of class  $C^K$  ( $0 \leq k \leq \omega$ ) and the maps  $((x, y) \longrightarrow x \times y), ((x, y) \longrightarrow x \setminus y)$  also of the class  $C^K$  then  $Q$  is called left local loop of class  $C^K$ . If at the place of  $\mathcal{U}$  consider all the space  $Q$ , then  $Q$  is called left topological loop or correspondently left loop of smoothness  $C^k$ .

**Definition I.3.1.** We will say that  $Q$  is a local Bol loop of class  $C^K$  ( $0 \leq k \leq \omega$ ), if  $Q$  is a manifold of class  $C^K$  with a fixed element  $e$  and for any neighborhood  $\mathcal{U}$  of the element  $e$  are defined the maps of class  $C^k$   $\mathcal{U} \ni x \longrightarrow x^{-1}$  and  $\mathcal{U} \times \mathcal{U} \ni (x, y) \longrightarrow x \cdot y \in Q$  verifying the conditions

- if  $x \in \mathcal{U}$  then  $e \cdot x = x \cdot e = x$
- if  $x, y, x \cdot y \in \mathcal{U}$  then  $x^{-1} \cdot x = x \cdot x^{-1} = x, x^{-1} \cdot (x \cdot y) = y$
- if  $x, y, z, x \cdot z, x \cdot y, x \cdot yx, y \cdot xz \in \mathcal{U}$  then the left Bol identity  $(x \cdot yx)z = x(y \cdot xz)$  is verify.

**Definition I.3.2.** Let  $V$  be a finite dimensional vector space where, it's defined a bilinear and a trilinear operation  $x \cdot y$  and  $\langle x, y, z \rangle$ . We will say that  $V$  is called a  $W$ -algebra if the following identities are verify:

1. 4.  $x \cdot x = 0$
2. 5.  $\langle x, x, x \rangle = 0$
3. 6.  $xy \cdot z + yz \cdot x + zx \cdot y = \langle x, y, z \rangle + \langle y, z, x \rangle + \langle z, x, y \rangle - \langle y, x, z \rangle - \langle z, y, x \rangle - \langle x, z, y \rangle.$

The operation of composition of any local loop  $(Q, \times, e)$  of class  $C^3$  has in the neighborhood of the unity element  $e$  the following coordinates expression (with accuracy up to the 3 order)

$$(x \times y)^i = x^i + y^i + \tau_{jk}^i x^j y^k + \mu_{jkl}^i x^j x^k y^l + \nu_{jkl}^i x^j y^k y^l + \dots$$

from the basic tensor of the local loop

$$\alpha_{jk}^i = \tau_{[jk]}^i$$

$$\beta_{jkl}^i = 2\mu_{jkl}^i - 2\nu_{jkl}^i + \alpha_{jk}^m \alpha_{mk}^i - \alpha_{jm}^i \alpha_{ik}^m$$

equipping the tangent space  $V = T_e(Q)$  with the composition law

$$[x, y]^i = 2\alpha_{jk}^i x^j y^k,$$

$$(x, y, z)^i = \beta_{jkl}^i x^j y^k z^l.$$

If  $\alpha(t)$ ,  $\beta(t)$ ,  $\gamma(t)$  be smooth curves in loop  $Q$  at  $t = 0$  going through the point  $e$  with tangent vectors  $x$ ,  $y$ ,  $z$  accordingly then

$$(\beta(t) \times \alpha(t)) \setminus (\alpha(t) \times \beta(t)) = t^2[x, y] + 0(t^2)$$

$$[\alpha(t) \times (\beta(t) \times \gamma(t))] \setminus [(\alpha(t) \times \beta(t)) \times \gamma(t)] = t^3 \langle x, y, z \rangle + 0(t^3)$$

The tensors  $\alpha_{jk}^i$  and  $\beta_{jkl}^i$  ( or what equivalent the operation  $[\cdot, \cdot], \langle \cdot, \cdot, \cdot \rangle$ ) are defining the  $C^w$  Bol loop with local isomorphism accuracy under the fulfillment of conditions:

- 4'.  $[x, x] = 0$
- 5'.  $\langle x, x, y \rangle = 0$
- 6'.  $[[x, y], z] + [[y, z], x] + [[z, x], y] = 2 \langle x, y, z \rangle + 2 \langle y, z, x \rangle + 2 \langle z, x, y \rangle$
7.  $\langle x, y, z, \eta \rangle - \langle x, y, \eta, z \rangle + \langle [x, y], \eta, z \rangle - \langle [x, y], z, \eta \rangle + \langle x, y, [\eta, z] \rangle = 0,$
8.  $(x, y, (z, \xi, \eta)) = ((x, y, z), \xi, \eta) + (z, (x, y, \xi), \eta) + (z, \xi, (x, y, \eta)),$

where  $(x, y, z) = 2 \langle x, y, z \rangle - [[x, y], z]$ .

**Definition I.3.3.** [46]  $W$ -algebra  $V$ , the basic operations for which the following conditions are satisfied:

- 5''.  $\langle x, x, y \rangle = 0$
- 6''.  $xy \cdot z + yz \cdot x + zx \cdot y = 2 \langle x, y, z \rangle + 2 \langle y, z, x \rangle + 2 \langle z, x, y \rangle$

$$7''. < x, y, z > \cdot \xi - < x, y, \xi > \cdot z + < x \cdot y, \xi, z > - < x \cdot y, z, \xi > + < x, y, \xi \cdot z > = 0$$

$$8. (x, y, (z, \xi, \eta)) = ((x, y, z), \xi, \eta) + (z, (x, y, \xi), \eta) + (z, \xi, (x, y, \eta)),$$

where  $(x, y, z) = -2 < x, y, z > + xy \cdot z$ .

Will be call a Bol algebras.

The system of the identities 5'', 6'', 7'' ' can be rewritten in the equivalent form:

$$9. (x, x, y) = 0$$

$$10. (x, y, z) + (y, z, x) + (z, x, y) = 0$$

$$11. (x, y, z) \cdot \xi - (x, y, \xi) \cdot z + (z, \xi, x \cdot y) - (x, y, z \cdot \xi) + xy \cdot z\xi = 0$$

here, one can consider an arbitrary binary-ternary algebra, of finite dimension with main operation for which  $x \cdot y$  and  $(x, y, z)$  satisfy the identities (8)-(10) as a Bol algebra. This point of view is more preferable because in this case one can regard the conditions (8)-(11) as the identities defining a Lie triple systems with the composition law  $(x, y, z)$ .

Let  $\langle Q, \cdot, e \rangle$  be a loop with the left Bol identity and  $G = \langle LQ \rangle$  his canonical enveloping group for the Bol loop  $Q$  and the subgroup  $H$  which correspond to the Lie group  $\mathfrak{G}$  and his subalgebra  $\mathfrak{h}$ . In the point of the local left coset section of the class  $G \bmod H$ .  $Q = \exp \mathcal{U}$  where  $\mathcal{U}$  is sufficiently small neighborhood of the 0 in  $\mathfrak{B}$  (Bol algebra). let's introduce the law of composition  $\times$ :  $x \times y = (x/e)y$  in result we obtain a loop  $\langle Q, \times, e \rangle$  with left Bol identity and the  $W$ -algebra is isomorphic to the to the original Bol algebra.

Let the local section  $B = \exp \mathcal{U}$  of left coset class  $G \bmod H$  verifying the conditions stated above  $x \in B$  and  $\tilde{H} = x\Delta H\Delta x^{-1}$  for  $x$  sufficiently near to  $e$ , we have  $\tilde{H} \cap B = \{e\}$  that is way in  $B$  one can define a new law of composition  $a \top b = \tilde{\Pi}_B(a\Delta b)$  where  $\tilde{\Pi}_B$  is the projection on  $B$  parallel to  $H$ . Since  $Q$  is the section  $G/H$  and  $f = x^2$  then  $(Q, \top)$  is obtain by the rotation  $x \cdot H \cdot x^{-1}$ . For  $y$  relatively near to  $e$  let's consider the operation:

$$a \perp b = (a/y) \times (y \times b).$$

According to Robinson [93], any arbitrary loop isotopic to the loop  $(Q, \times, e)$  is isomorphic to the loop  $(Q, \perp, e)$  for any  $y$  see [79-80].

**Theorem I.31.1** Local analytic Bol loops  $B$  and  $B'$  are local isomorphic if and only if their correspondent Bol algebras  $V$  and  $V'$  are isomorphic.

**Theorem I.3.2** An arbitrary Bol algebra is a  $W$ -algebra of an analytic Bol loop.

## 2.5 IMBEDDING OF A LOCAL ANALYTIC BOL LOOP INTO A LOCAL LIE GROUP

Let  $(G, \Delta, e)$ - be a local Lie group and  $H$ - be one of its subgroups, and let's denoted the corresponding Lie algebra and subalgebra by  $\mathfrak{G}$  and  $\mathfrak{h}$ . Consider

a vector subspace  $\mathfrak{B}$  such that

$\mathfrak{G} = \mathfrak{h} \dot{+} \mathfrak{B}$ . Let  $\Pi : G \longrightarrow G \setminus H$  be the canonical projection and let  $\Psi$  be the restriction of mappings composition  $\Pi \circ \exp$ , to  $\mathfrak{B}$ . Then there exists such a neighborhood  $\mathcal{U}$  of the point  $O$  in  $\mathfrak{B}$  such that,  $\Psi$  maps it diffeomorphically into the neighborhood  $\Psi(u)$  of the coset  $\Pi(e)$  in  $G \setminus H$  [46].

$$\begin{array}{ccc} \mathfrak{G} & \xleftarrow{i} & \mathfrak{B} \\ \exp \downarrow & & \downarrow \Psi \\ G & \xrightarrow{\pi} & G \setminus H \end{array}$$

Introduce a local composition law:

$$a \star b = \Pi_B(a \Delta b)$$

On points of local cross-section  $B = \exp \mathcal{U}$  of left cosets of  $G \bmod H$ , where  $\Pi_B = \exp \circ \Psi^{-1} \circ \Pi : G \longrightarrow B$  is the local projection on  $B$  parallel to  $H$  which, puts into correspondence every element  $a \in B$  so that  $g = a \Delta p$ , where  $p \in H$ .

**Proposition I.3.1**[46] Let's assume that for any  $a, b \in B = \exp \mathcal{U}$  sufficiently close to the point  $e$ , and  $a \Delta b \Delta a \in B$ ; then the local analytic loop  $(B, \times, e)$  satisfies the left Bol condition.

## 2.6 IMBEDDING OF BOL ALGEBRA INTO LIE ALGEBRAS: ENVELOPING LIE ALGEBRA OF BOL ALGEBRAS

Let the local cross-section  $B = \exp \mathcal{U}$  of left cosets  $G \setminus H$  satisfy the condition of Proposition I.3.1 above. It is interesting to calculate the, operations of the Bol algebra  $\mathfrak{B}$  tangent to the local analytic Bol loop  $(B, \star, e)$  in terms of a Lie algebra  $\mathfrak{G}$ , its subalgebra  $\mathfrak{h}$  and vector subspace  $\mathfrak{B}$ . Introduce in  $G$  normal coordinates then:

$$a \star b = a + b + \frac{1}{2}[a, b]_{\mathfrak{B}} + 0(2)$$

$\forall a, b \in B$  where  $[a, b]_{\mathfrak{B}}$  is the projection of  $[a, b]$  on  $\mathfrak{B}$  parallel to  $\mathfrak{h}$  thus

$$\xi \cdot \eta = [\xi, \eta]_{\mathfrak{B}}$$

$$(\xi, \eta, \chi) = [[\xi, \eta], \chi]$$

$$\langle \xi, \eta, \chi \rangle = -\frac{1}{2}[[\xi, \eta], \chi] + \frac{1}{2}[[\xi, \eta]_{\mathfrak{B}}, \chi]_{\mathfrak{B}}.$$

One can find the correctness of the following propositions [30,44]:

**Proposition I.3.2.** [46] Any local analytic Bol loops  $B$  and  $B'$  are locally isomorphic, if and only if their corresponding Bol algebras  $\mathfrak{B}$  and  $\mathfrak{B}'$  are isomorphic.

**Proposition I.3.3.** [46] An arbitrary Bol algebra is a tangent  $W$ -algebra of some local analytic loop with the left Bol loop.

**Proposition I.3.4.** [46] The property  $a\Delta b\Delta a \in B$  is held for any  $a, b \in B$  sufficiently close to  $e$ , if and only if  $[[\xi, \eta], \zeta] \in \mathfrak{B} \forall \xi, \eta, \zeta \in \mathfrak{B}$ .

Let  $\mathfrak{B}$  be a finite Bol algebra over  $\mathbb{R}$ , the basic operations of which are  $\xi \cdot \eta$  and  $(\xi, \eta, \zeta)$ ,  $\mathfrak{G}$ -finite Lie algebra over  $\mathbb{R}$ ,  $\mathfrak{h}$  - subalgebra of  $\mathfrak{G}$  and  $i: \mathfrak{B} \rightarrow \mathfrak{G}$  a linear mapping such that  $i(\mathfrak{B}) \in \mathfrak{B}$ .

$\mathfrak{G} \dot{+} \mathfrak{h}$  (direct sum of vector spaces)

$$[[\mathfrak{B}], \mathfrak{B}], \mathfrak{B}] \in \mathfrak{B} \text{ and } \forall \xi, \eta, \zeta \in \mathfrak{B}$$

$$\xi \cdot \eta = [\xi, \eta]_{\mathfrak{B}} \quad (\xi, \eta, \zeta) = [[\xi, \eta], \zeta]$$

where  $[\xi, \eta]$  denotes the result of commutation of vectors in  $\mathfrak{G}$  and  $[\xi, \eta]_{\mathfrak{B}}$  denotes projection of the vector  $[\xi, \eta]$  on  $\mathfrak{B}$  parallel to  $\mathfrak{h}$ .

In that case we will talk about the enveloping pair  $(\mathfrak{G}, \mathfrak{h})$  Lie algebra of Bol algebra  $\mathfrak{B}$  or, in other words enveloping Lie algebra  $\mathfrak{G}$  of Bol algebra  $\mathfrak{B}$ .

Let  $(\mathfrak{G}, \mathfrak{h})$  be an enveloping pair of Lie algebra of Bol algebra  $\mathfrak{B}$ . Let us identify  $\mathfrak{B}$  with a vector subspace  $i(\mathfrak{B})$  into  $\mathfrak{G}$ , and let us consider the subalgebra  $\mathfrak{G}' = \mathfrak{B} \dot{+} [\mathfrak{B}, \mathfrak{B}]$  into  $\mathfrak{G}$ , and the subalgebra  $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{G}'$  into  $\mathfrak{G}'$ , then the pair  $(\mathfrak{G}', \mathfrak{h}')$  is also for enveloping for a Bol algebra  $\mathfrak{B}$ .

By the construction of the Lie algebra which is a canonical enveloping for  $\mathfrak{B}$ , it is better to use the construction made in [45]. For such Bol algebra  $\mathfrak{B}$ , there exists a Lie algebra  $\tilde{\mathfrak{G}}$ , and an enveloping automorphism  $\tau \in \text{Aut} \tilde{\mathfrak{G}}$  such that  $(\tau^2 = Id)$ , linear injection map  $i: \mathfrak{B} \rightarrow \tilde{\mathfrak{G}}$  and a subalgebra  $\tilde{\mathfrak{h}}$  in  $\tilde{\mathfrak{G}}$ , such that ( we are identifying  $i(\mathfrak{B})$  with  $\mathfrak{B}$ ).

$$\tilde{\mathfrak{G}} = \tilde{\mathfrak{G}}_- + \tilde{\mathfrak{G}}_+, \text{ where } \tilde{\mathfrak{G}}_- = \mathfrak{B}; \tilde{\mathfrak{G}} = \mathfrak{B} \dot{+} \tilde{\mathfrak{h}}, < \mathfrak{B} > = \tilde{\mathfrak{G}}$$

$$\xi \cdot \eta = \prod_{\mathfrak{B}} [\xi, \eta] = [\xi, \eta]_{\mathfrak{B}}$$

$$(\xi, \eta, \zeta) = [[\xi, \eta], \zeta]$$

Lie algebra  $\tilde{\mathfrak{G}}$  is an enveloping Lie algebra for Bol algebra  $\mathfrak{B}$ , but in general not canonical enveloping because  $\mathfrak{h}$  may contain an ideal  $I$  of Lie algebra  $\tilde{\mathfrak{G}}$ . Hence the canonical enveloping Lie algebra  $\mathfrak{G}$  for a Bol algebra  $\mathfrak{B}$  is obtained by factorizing  $\tilde{\mathfrak{G}}$  with the ideal  $I$ . Therefore after the factorization of  $\tilde{\mathfrak{G}}_+ \setminus I$  and  $\mathfrak{B}$  ( we identify  $\mathfrak{B}$  and  $\mathfrak{B} \setminus I$ ) in general interest, let us note that the construction see [45] follows that  $\dim \tilde{\mathfrak{G}} \leq \dim \mathfrak{B} \wedge \mathfrak{B} + \dim \mathfrak{B}$ .

In our case  $\dim \mathfrak{B} = 3$ , that is why under the examination of the corresponding canonical enveloping Lie algebra  $\mathfrak{G}$  we must consider the case:

$$\dim[\mathfrak{B}, \mathfrak{B}] = 0, 1, 2, 3.$$

## 2.7 THREE-WEBS COORDINATES LOOP OF THREE-WEBS

**Definition I.4.1** Let  $W$  be a  $C^\infty$  smooth manifold of  $2N$  dimension, equipped with such foliation  $\lambda_i$ ,  $i = 1, 2, 3$  codimension  $N$ , for any point from  $W$ , then there exists a neighborhood  $\mathcal{U}$  containing a point, such that the fiber of any two different foliations have in  $\mathcal{U}$  not more than one common point. In that case  $(W, \lambda_1, \lambda_2, \lambda_3)$  is called local three-web. The three-web  $(W, \lambda_1, \lambda_2, \lambda_3)$  is called global, if for any two fiber from the different families  $\lambda_i$  and  $\lambda_j$  ( $i, j = 1, 2, 3, i \neq j$ ) have exactly one common point.

About the three-web theory see [2, 3, 4, 5, 6, 9].

Let  $P$  be a fixed point of the three-Web  $W$  and let  $\mathcal{F}_a \in \lambda_1$  and  $\mathcal{F}_b$  two fibers of web, passing through  $P$  and let  $q(a, b) = c \in x_3$ . Let us examine the map:

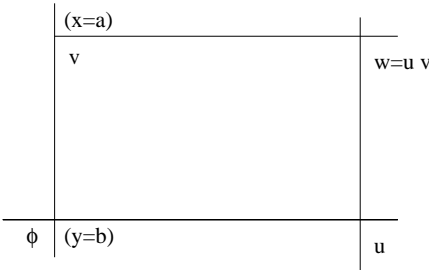
$$u = q(x, b), v = q(a, y), z = z. \quad (1)$$

The foliations  $\lambda_1, \lambda_2$ , abide us to define, the operation of multiplication in  $\lambda_3$  such that:

$$u \cdot v = q(x, y) = q(q^{-1}(u, b), q^{-1}(a, v))$$

where  $c$  denotes the unity.

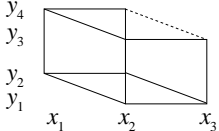
If the neighborhood  $\mathcal{U} \ni p$  is so small, the map (1) is a bijection which defines some isotopic transformation of quasigroup  $q$ . The so obtained isotope, is called principal isotope. It's having a neutral element that is why it's a loop  $l(a, b)$ .


(3)

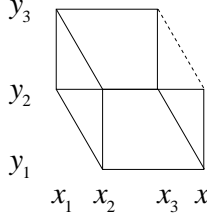
### CONDITIONS FOR CLOSURES

In the three-web theory, the condition for closure of some figures formed by the web and point in their intersection plays an important role.

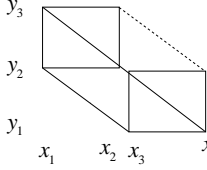
Let us examine a three-web  $W$  and let  $z = q(x, y)$  it's coordinate loop. It is said that on the web  $W$  the figure of closure verifies one of the following types if:



$$\left. \begin{aligned} q(x_1, y_2) &= q(x_2, y_1) \\ q(x_2, y_2) &= q(x_3, y_1) \\ q(x_1, y_4) &= q(x_2, y_3) \end{aligned} \right\} \longrightarrow q(x_2, y_4) = q(x_3, y_3) \quad (4)$$



$$\left. \begin{aligned} q(x_1, y_2) &= q(x_2, y_1) \\ q(x_2, y_2) &= q(x_1, y_3) \\ q(x_4, y_1) &= q(x_3, y_2) \end{aligned} \right\} \longrightarrow q(x_3, y_3) = q(x_4, y_2) \quad (5)$$



$$\left. \begin{aligned} q(x_1, y_2) &= q(x_2, y_1) \\ q(x_1, y_4) &= q(x_2, y_3) \\ q(x_3, y_2) &= q(x_4, y_1) \end{aligned} \right\} \longrightarrow q(x_3, y_4) = q(x_4, y_3) \quad (6)$$

These figures are respectively called left, right and mean Bol figure and it is denoted by  $B_l$ ,  $B_r$ ,  $B_m$ .

**Proposition I.4.1.** [4] Three-web  $W$  is a Bol web if any of its coordinate loops is a Bol loop. Any local analytical loop can be realized as some coordinate loop of a certain three-web.

**Proposition I.4.2** [4]

1. The coordinate loop of a global three-web is an isotope.
2. The three-web, corresponding (analytical) to isotopic (global) loops are isomorphic.

## 2.8 ISOTOPIC BOL ALGEBRAS

Below we give a correct generalization of the notion of isotopy of (global) loops to the case of local analytic Bol loops. Our approach is based on a construction of embedding of Bol loops into a group and on an interpretation of isotopic loops in terms of their enveloping groups [48,86].

**Definition I.5.1.** Let  $\mathfrak{B}$  and  $\tilde{\mathfrak{B}}$  be two Bol algebras and let  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}$  and  $\tilde{\mathfrak{G}} = \tilde{\mathfrak{B}} \dot{+} \tilde{\mathfrak{h}}$  be their canonical enveloping Lie algebras. The algebras  $\mathfrak{B}$  and  $\tilde{\mathfrak{B}}$  will be called isotopic if there exist such a Lie algebras isomorphism  $\Phi : \mathfrak{G} \longrightarrow \tilde{\mathfrak{G}}$ , such that  $\Phi(\mathfrak{G}) = \tilde{\mathfrak{G}}$  and  $\Phi(\mathfrak{h})$  coincide with the image of the subalgebra  $\tilde{\mathfrak{h}}$  in  $\tilde{\mathfrak{G}}$  under the action of an inner automorphism  $Ad\xi$ ,  $\xi \in \tilde{\mathfrak{G}}$  i.e.

$$\Phi(\mathfrak{h}) = (Ad_\xi)\tilde{\mathfrak{h}}$$

It is clear that the notion of isotopy is not an equivalence relation to Bol algebra manifolds.

**Theorem I.5.1** Let  $B(\times)$  and  $\tilde{B}(\circ)$  be global analytic Bol loops, and let their tangent Bol algebras be isotopic, then  $\tilde{B}(\circ)$  is locally isomorphic to an analytic Bol loop analytically isotopic to  $B(\times)$ .

Proof.

Let  $\mathfrak{G}$  be the canonical Lie algebra enveloping the Bol algebra  $\mathfrak{B}$  tangent to



the analytic Bol loop  $B(\times)$ .  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}$  ( direct sum of vector spaces). There exists a Lie group  $G$  with  $\mathfrak{G}$  as its tangent Lie algebra, a closed subgroup  $G_O$  corresponding to the subalgebra  $\mathfrak{h}$  and an analytical embedding  $i : B \longrightarrow G$  such that the composition law takes the following form [48,86]

$$a \times b = \prod(a, b),$$

where  $a \times b$  denotes the composition of elements  $a, b$  in  $B$  and  $\prod : G \longrightarrow B$  is the projection on  $B$  parallel to  $G_O$ .

Let us suppose that the Bol algebra  $\mathfrak{B}$  tangent to the loop  $\tilde{B}(\circ)$  is isotopic to the Bol algebra  $\mathfrak{B}$ , i.e. there exist Lie algebra isomorphism  $\Phi : \mathfrak{G} \longrightarrow \mathfrak{G}$ , such that  $\Phi(\mathfrak{G}) = \mathfrak{G}$  and  $\Phi(\mathfrak{h})$  coincide with the image of the subalgebra  $\mathfrak{h}$  in  $\mathfrak{G}$  under the action of an inner automorphism  $Ad\xi$ ,  $\xi \in \mathfrak{B}$ ,  $\Phi(\mathfrak{h}) = (Ad\xi)\mathfrak{h}$ .

Let us introduce the element  $y = \exp(\xi)$  and the subgroup  $\tilde{G}_O = yG_Oy^{-1}$ . The analytic loop  $B(\star)$

$$a \star b = \widetilde{\prod}(ab), a, b \in B$$

where  $\widetilde{\prod} : G \longrightarrow B$  is the projection on  $B$  parallel to  $\tilde{G}_O$ , is a Bol loop whose tangent Bol algebra is isomorphic to  $\mathfrak{B}$ . In particular,  $B(\star)$  and  $\tilde{B}(\circ)$  are locally isomorphic. The operations on  $B$  are isotopic. Indeed let us introduce an analytic diffeomorphism  $\Omega : B \longrightarrow B$ ,  $a \longmapsto (y \times a) \setminus y$ , then:

$$\begin{aligned} \Omega_y^{-1}(\Omega_y(a) \star \Omega_y(b)) &= (L_y)^{-1}(\Omega_y(a) \times L_y b) = Y^{-1} \times [(y \times a) \setminus y] \times (y \times b) \\ &= [y^{-1} \times ((y \times a) \setminus y) \times y^{-1}] \times (y^2 \times b) \\ &= [y^{-1} \times ((y^{-1} \times ((y^2 \times a) \times y^{-1})))] \times (y^2 \times b) \\ &= [y^{-1} \times [y^{-1} \times ((y^2 \times a) \times y^{-2})]] \times (y^2 \times b) \\ &= [y^{-2} \times ((y^2 \times a) \times y^{-2})] \times (y^2 \times b) \\ &= (a \setminus y^2) \times (y^2 \times b) \\ &= (a \setminus y) \times (y \times b) \end{aligned}$$

therefore  $\Omega_y((a \setminus y) \times (y \times b)) = \Omega_y(a) \star \Omega_y(b)$ . hence  $B(\times)$  is isotopic to  $\tilde{B}(\circ)$  and by the diffeomorphism  $\Omega$  they are isomorphic. Hence the theorem is proved.

## 2.9 ABOUT THE CLASSIFICATION OF BOL ALGEBRAS

Let  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}$  be Lie algebra in the involutive decomposition  $\mathfrak{h}'$  a subalgebra in  $\mathfrak{G}$  such that  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}'$

$$a \cdot b = \prod[a, b]$$

$$(a, b, c) = [[a, b], c]$$

where  $\Pi : \mathfrak{G} \longrightarrow \mathfrak{B}$  is a projection on  $\mathfrak{B}$  parallel to  $\mathfrak{h}'$  and  $[\cdot]$  commutator in Lie algebra  $\mathfrak{G}$ .

That's why the classification of Bol algebra lead to the classification of subalgebra  $\mathfrak{h}'$ , in the enveloping Lie algebra  $\mathfrak{G}$  (not necessarily canonical) for a Lie triple system  $\mathfrak{M}$ .

Below, we will examine the classification of Bol algebras with isomorphism accuracy and isotopic accuracy [66]. The classification with isotopic accuracy is more crude, than the classification with isomorphism accuracy. However the notion of isotopy of Bol algebra is opening a new connection between non isomorphic Bol algebras.

## 2.10 ISOCLINE BOL ALGEBRAS

One can prove that any vectorial space  $\mathfrak{B}$ , equipped with operation

$$\forall \xi, \eta, \zeta \in \mathfrak{B} \quad \xi \cdot \eta = \alpha(\xi)\eta - \alpha(\eta)\xi,$$

$$< \xi, \eta, \zeta > = \beta(\xi, \zeta)\eta - \beta(\eta, \zeta)\xi \quad (I)$$

where  $\alpha : \mathfrak{B} \longrightarrow \mathbb{R}$ -linear form,  $\beta : \mathfrak{B} \times \mathfrak{B} \longrightarrow \mathbb{R}$ -bilinear symmetric form. Is a Bol algebra.

**Definition I.7.** Bol algebra of view (I) is called isocline.

One can prove see [46], that any Bol algebra is called isocline if and only if it verifies the plane axiom that means any two-dimensional vectorial subspace (plane) is a subalgebra.

In particular any two-dimensional Bol algebra is isocline.

If  $\dim \mathfrak{B} = 3$  and  $\alpha = 0$ , then depending on the rank, and the signature of the form  $\beta$ , we obtained 5 non trivial and non isomorphic types of Lie triple systems.

### 3 CHAPTER II

#### CLASSIFICATION OF SOLVABLE 3-DIMENSIONAL LIE TRIPLE SYSTEMS EXAMPLE OF 3-DIMENSIONAL BOL ALGEBRAS

##### 3.1 ABOUT LIE TRIPLE SYSTEMS

**Definition II.1.1** The vector space  $\mathfrak{M}$  (finite over the field of real numbers  $\mathbb{R}$ ) with trilinear operation  $(x, y, z)$  is called Lie triples system if:

$$(x, x, y) = 0$$

$$(x, y, z) + (y, z, x) + (z, x, y) = 0$$

$$(x, y, (u, v, w)) = ((x, y, u), v, w) + (u, (x, y, v), w) + (u, v, (x, y, w))$$

Let  $\mathfrak{M}$  be a Lie triple system, the subspace  $\mathfrak{D} \subset \mathfrak{M}$  is called subsystem if  $(\mathfrak{D}, \mathfrak{D}, \mathfrak{D}) \subset \mathfrak{D}$ , and is ideal, if  $(\mathfrak{D}, \mathfrak{M}, \mathfrak{M}) \subset \mathfrak{D}$ . The ideals are the Kernel of the homomorphism of the Lie triple system [26,52].

**Example** For a typical way of construction of a Lie triple system see in [26,52].

Let  $\mathfrak{G}$  be a Lie algebra (finite over the field of real numbers  $\mathbb{R}$ ) and  $\sigma$ -an involutive automorphism, then

$$\mathfrak{G} = \mathfrak{G}^+ \dot{+} \mathfrak{G}^-$$

where  $\sigma|_{\mathfrak{G}^+} = Id$  and  $\sigma|_{\mathfrak{G}^-} = -Id$ , as for any element  $x$  from  $\mathfrak{G}$  can be written in the form:

$$x = \frac{1}{2}(x + \sigma x) + \frac{1}{2}(x - \sigma x),$$

where  $x + \sigma x \in \mathfrak{G}^+$ ,  $x - \sigma x \in \mathfrak{G}^-$  and  $\mathfrak{G}^+ \cap \mathfrak{G}^- = 0$ .

The following inclusions are held:

$$[\mathfrak{G}^+, \mathfrak{G}^+] \subset \mathfrak{G}^+, [\mathfrak{G}^+, \mathfrak{G}^-] \subset \mathfrak{G}^-, [\mathfrak{G}^-, \mathfrak{G}^-] \subset \mathfrak{G}^+.$$

Such that the subspace  $\mathfrak{G}^-$  turns into a Lie triple system relatively under the operation  $(x, y, z) = [[x, y], z]$ .

The inverse construction [26].

Let  $\mathfrak{M}$  be a Lie triple system and define by

$$h(X, Y) : z \longrightarrow (X, Y, Z)$$

a linear transformation of the space  $\mathfrak{M}$  into it selves where  $X, Y, \in \mathfrak{M}$ .

Let  $H$  be a subspace in the space of linear transformations, of Lie triple systems  $\mathfrak{M}$  of the transformations of the form  $h(X, Y)$ . The vector space  $\mathfrak{G} = \mathfrak{M} \dot{+} H$ , become a Lie algebra relatively to the commutator  $[A, B] =$

$AB - BA, [A, X] = -[X, A] = AX; [X, Y] = h(X, Y)$  where  $A, B \in H, X, Y \in \mathfrak{M}$ .

Let define the mapping  $\sigma$  with the condition  $\sigma(A) = A, A \in H$  and  $\sigma(X) = -X, X \in \mathfrak{M}$ , then  $\sigma$ - is an involutif automorphism of Lie algebra  $\mathfrak{G} = \mathfrak{M} \dot{+} H$ .

The algebra  $\mathfrak{G}$  constructed above from the Lie triple system, is called universal enveloping Lie algebra of the Lie triple system  $\mathfrak{M}$ .

**Definition II.1.2** The derivation of the Lie triple system  $\mathfrak{M}$ , is called the linear transformation  $\mathfrak{d} : \mathfrak{M} \longrightarrow \mathfrak{M}$  such that

$$(X, Y, Z)\mathfrak{d} = (X\mathfrak{d}, Y, Z) + (X, Y\mathfrak{d}, Z) + (X, Y, Z\mathfrak{d}).$$

One can verify that, the set  $\mathfrak{d}(\mathfrak{M})$  of all the derivation of the Lie triple systems  $\mathfrak{M}$  is a Lie algebra of the linear transformations acting on  $\mathfrak{M}$ .

**Definition II.1.3** The imbedding of a Lie triple system  $\mathfrak{M}$  into a Lie algebra  $\mathfrak{G}$  is called the linear injection  $R : \mathfrak{M} \longrightarrow \mathfrak{G}$  such that  $(X, Y, Z) = [[X^R, Y^R], Z^R]$ .

The imbedding  $R$  of the Lie triple system  $\mathfrak{M}$  into the Lie algebra  $\mathfrak{G}$  is called canonical, if the envelope of the image of the set  $\mathfrak{M}^R$  in the Lie algebra  $\mathfrak{G}$  coincide with  $\mathfrak{G}$  and  $h$  does not contain trivial ideals of Lie algebra  $\mathfrak{G}$ . Let us note that if the Lie triple system  $\mathfrak{M}$  is a subset of the Lie algebra  $\mathfrak{G}$ , then  $(X, Y, Z) = [[X, Y], Z]$  and  $[\mathfrak{M}, \mathfrak{M}]$  is a subalgebra of the Lie algebra  $\mathfrak{G}$  hence  $\mathfrak{M} + [\mathfrak{M}, \mathfrak{M}]$  is a Lie subalgebra of  $\mathfrak{G}$  and the initial imbedding  $R$  can be consider as canonical in  $\mathfrak{M}^R + [\mathfrak{M}^R, \mathfrak{M}^R]$ ; this lead us to formulate the following proposition:

**Proposition II.1.1** For any finite Lie triple system  $\mathfrak{M}$  over  $\mathbb{R}$ , there exist one and only up to automorphism accuracy, one canonical imbedding to the Lie algebra.

## SOLVABLE AND SEMISIMPLE LIE TRIPLE SYSTEM

Following [85]:, let  $\Omega$ - be an ideal of the Lie triple system  $\mathfrak{M}$ , we assume  $\Omega^{(1)} = (\mathfrak{M}, \Omega, \Omega)$  and,  $\Omega^{(k)} = (\mathfrak{M}, \Omega^{(k-1)}, \Omega^{(k-1)})$

**Proposition II.1.2** [85] For all natural number  $k$ , the subspace  $\Omega^{(k)}$  is an ideal of  $\mathfrak{M}$  and we have the following inclusions:

$$\Omega \supseteq \Omega^{(1)} \supseteq \dots \supseteq \Omega^{(k)}$$

Proof

$$(\Omega^{(1)}, \mathfrak{M}, \mathfrak{M}) = ((\mathfrak{M}, \Omega, \Omega), \mathfrak{M}, \mathfrak{M}) \subseteq ((\mathfrak{M}, \Omega, \mathfrak{M}), \Omega, \mathfrak{M}) + [[[\mathfrak{M}, \Omega], [\mathfrak{M}, \Omega]], \mathfrak{M}]$$

according to the definition of a Lie triple system

$$(\Omega^{(1)}, \mathfrak{M}, \mathfrak{M}) \subseteq (\Omega, \Omega, \mathfrak{M}) + [(\mathfrak{M}, \Omega, \mathfrak{M}), [\mathfrak{M}, \Omega]] \subseteq (\mathfrak{M}, \Omega, \Omega) + (\mathfrak{M}, \Omega, \Omega) = \Omega^{(1)}$$

that means  $\Omega^{(1)}$  is an ideal of  $\mathfrak{M}$  further more  $\Omega^{(k)} = (\Omega^{(k-1)})^{(1)}$  hence each  $\Omega^{(i)}$  is an ideal in  $\mathfrak{M}$ .

**Definition II.1.4** The ideal  $\Omega$  of a Lie triple system  $\mathfrak{M}$  is called solvable, if there exist a natural number  $k$  such that  $\Omega^{(k)} = 0$ .

**Proposition II.1.3** [85] If  $\Omega$  and  $\Theta$  are two solvable ideals of a Lie triple system  $\mathfrak{M}$  then  $\Omega + \Theta$  is also a solvable ideal in  $\mathfrak{M}$ .

Proof

using the definition of a Lie triple system, the following inclusion hold:  
 $(\Theta + \Omega)^{(1)} \subseteq (\mathfrak{M}, \Theta, \Theta) + (\mathfrak{M}, \Omega, \Omega) + (\mathfrak{M}, \Theta, \Omega) + (\mathfrak{M}, \Omega, \Theta) \subseteq \Theta^{(1)} + \Omega^{(1)} + \Theta \cap \Omega$ .

Assume for every natural number  $k$  the following inclusion holds:

$$(\Theta + \Omega)^{(k)} \subseteq \Theta^{(k)} + \Omega^{(k)} + \Theta \cap \Omega$$

by induction let's prove that its holds for  $(k+1)$

$$(\Theta + \Omega)^{(k+1)} = (\mathfrak{M}, (\Theta + \Omega)^{(k)}, (\Theta + \Omega)^{(k)}) \subseteq (\mathfrak{M}, \Theta^{(k)} + \Omega^{(k)} + \Theta \cap \Omega, (\Theta \cap \Omega)) \subseteq \Theta^{(k+1)} + \Omega^{(k+1)} + \Theta \cap \Omega$$

hence the result

**Definition II.1.5** The radical of a Lie triple system denoted by  $\mathfrak{R}(\mathfrak{M})$ , is called the maximal solvable ideal of the Lie triple system  $\mathfrak{M}$ .

A Lie triple system  $\mathfrak{M}$  is called semi-simple if  $\mathfrak{R}(\mathfrak{M}) = 0$ .

**Theorem II.1.1** [85] If  $\mathfrak{R}$  is a radical in  $\mathfrak{M}$  then  $(\mathfrak{M} \setminus \mathfrak{R})$  is semisimple. And if  $\Omega$  is an ideal in  $\mathfrak{M}$  such that  $(\mathfrak{M} \setminus \mathfrak{R})$  is semisimple then  $\Omega \supset \mathfrak{R}$ .

**Proposition II.1.4** [85] The enveloping Lie algebra, of a solvable Lie triple system is solvable. And if a Lie triple system has some solvable enveloping Lie algebra, it is solvable.

**Theorem II.1.2** If  $\mathfrak{M}$  is a semisimple Lie triple system, then the universal enveloping Lie algebra  $\mathfrak{G}$  is semisimple.

**Theorem II.1.3** Let  $\mathfrak{M}$  be a Lie triple system and  $\mathfrak{G} = \mathfrak{M} \dot{+} \mathfrak{h}$  his canonical enveloping Lie algebra and  $\mathfrak{r}$ - the radical of the Lie algebra  $\mathfrak{G}$ . In  $\mathfrak{G}$  there exist a subalgebra  $\mathfrak{P}$  semisimple supplementary to with  $\mathfrak{r}$  such that:

$$\mathfrak{M} = \mathfrak{M}' \dot{+} \mathfrak{M}'' \text{ (direct sum of vectors spaces)}$$

where

$$\mathfrak{M}' = \mathfrak{M} \cap \mathfrak{r} - \text{radical of the Lie triple system } \mathfrak{M}$$

$$\mathfrak{M}'' = \mathfrak{M} \cap \mathfrak{P} - \text{semisimple subalgebra of Lie triple system } \mathfrak{M}$$

$$\mathfrak{h} = \mathfrak{h}' \dot{+} \mathfrak{h}'' \text{ (direct sum of vectors spaces)}$$

$$\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{r}$$

and

$$\mathfrak{h}'' = \mathfrak{h} \cap \mathfrak{P} \text{ are subalgebra in } \mathfrak{h}$$

$$\mathfrak{r} = \mathfrak{M}' \dot{+} \mathfrak{h}'$$

$$\mathfrak{P} = \mathfrak{M}'' \dot{+} \mathfrak{h}''.$$

### 3.2 PROBLEM SETTING

Let  $\mathfrak{M}$  be a Lie triple system and  $\dim \mathfrak{M} = 3$ . To be consistent with the above Theorem the following cases are possible:

1. semisimple case  
 $\mathfrak{M}$ - semisimple Lie triple System (in fact simple). About the classification of such Lie triple systems see [85, 64, 53]
2. Splitting case

$$\mathfrak{M} = \mathfrak{M}_1 \dot{+} \mathfrak{M}_2$$

where  $\mathfrak{M} \equiv \mathbb{R}$ - solvable ideal of dimension 1 in  $\mathbb{R}$  and  $\mathfrak{M}_2$  -semisimple Lie triple system of dimension 2 This type of Lie triple system is not considered in this survey.

3. Solvable case  
 $\mathfrak{M}$  is a solvable Lie triple system. The classification of such system is given after the next paragraph.

#### CLASSIFICATION OF LIE TRIPLE SYSTEM OF DIMENSION 2

For a better survey of such Lie triple system, we will write their trilinear operation in a special form.

Let  $\mathfrak{M}$  be a 2-dimensional Lie triple system we write the trilinear operation  $(X, Y, Z) = \beta(X, Y)Y - \beta(Y, Z)X$  where  $\beta : V \times V \rightarrow \mathbb{R}$  is a symmetric form. The choice of the basis  $V = \langle e_1, e_2 \rangle$  one can reduce the symmetric form to the view:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \nu \end{pmatrix},$$

where  $\alpha, \nu = \pm 1; 0$ .

By introducing the notation of the derivation

$$\mathfrak{D}_{x,y} : \mathfrak{M} \longrightarrow \mathfrak{M}$$

$$z \longmapsto (x, y, z)$$

$$\mathfrak{h} = \{\mathfrak{D}_{x,y}\}_{x,y \in \mathfrak{M}}.$$

And

$\mathfrak{G} = \mathfrak{M} \dot{+} \mathfrak{h}$ - canonical enveloping Lie algebra of the Lie triple system  $\mathfrak{M}$ .

Let  $\mathfrak{M} = \langle e_1, e_2 \rangle$  then,

$$\mathfrak{h} = \{tD_{x,y}\}_{t \in \mathbb{R}},$$

$$e_1 D = (e_1, e_2, e_1) = \beta(e_1, e_1)e_2$$

$$e_2 D = (e_1, e_2, e_2) = -\beta(e_2, e_2)e_1$$

$$\mathfrak{G} = \langle e_1, e_2, e_3 \rangle$$

$$\text{where } [e_1, e_2] = e_3, [e_1, e_3] = -e_1 D, [e_2, e_3] = -e_2 D$$

Therefore we can have the up to isomorphism accuracy the following five cases:

1. (Spherical Geometry)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathfrak{G}/\mathfrak{h} \cong so(3)/so(2)$$

2. (Lobatchevski Geometry)

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathfrak{G}/\mathfrak{h} \cong sl(2, \mathbb{R})/so(2)$$

3. Lie triple system with non compact subalgebra  $\mathfrak{h}$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathfrak{G}/\mathfrak{h} \cong sl(2, \mathbb{R})/\mathbb{R}$$

4. Solvable case

- a)

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$e_1 \cdot e_2 = e_3, e_1 \cdot e_3 = e_2$$

(This is a Lie algebra  $\mathfrak{G}$  of type  $g_{3,5}(p=0)$  in [37])

- b)

$$\beta = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$e_1 \cdot e_2 = e_3, e_1 \cdot e_3 = -e_2$$

(This is a Lie algebra  $\mathfrak{G}$  of type  $g_{3,4}(h=-1)$  in [37])

5. Abelian case

$$\beta = 0 \quad \mathfrak{G}/\mathfrak{h} \cong (\mathbb{R})^2 / \{0\}$$

### 3.3 CLASSIFICATION OF SOLVABLE LIE TRIPLE SYSTEMS OF DIMENSION 3

Let  $\mathfrak{M}$  be a solvable Lie triple system of dimension 3, and  $\mathfrak{G} \wr \mathfrak{h}$  its canonical enveloping Lie algebra then  $\mathfrak{G}$  is solvable in particular  $\mathfrak{G}$  posses a characteristic ideal  $\mathfrak{G}' = [\mathfrak{G}, \mathfrak{G}] \triangleright \mathfrak{G}$ ,

$\sigma\mathfrak{G}' = \mathfrak{G}'$ ,  $\mathfrak{G}' \cap \mathfrak{M} = \mathfrak{M}' = (\mathfrak{M}, \mathfrak{M}, \mathfrak{M})$  further more  $\mathfrak{h} \subset \mathfrak{G}$  since  $\mathfrak{h} = [\mathfrak{M}, \mathfrak{M}]$  then

$$\mathfrak{G}' = [\mathfrak{G}, \mathfrak{G}] = \mathfrak{M}' + \mathfrak{h} \text{ where } \mathfrak{M}' \subsetneq \mathfrak{M}$$

Possible situations:

1.  $\dim \mathfrak{M}' = 0$ . Then  $[\mathfrak{h}, \mathfrak{M}] = \mathfrak{M}' = \{O\}$ , that means  $\mathfrak{h} \triangleright \mathfrak{G}$ - ideal, that is why  $\mathfrak{h} = \{O\}$  (since  $\mathfrak{G}$ - is an enveloping Lie algebra) and  $\mathfrak{M} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ . In this case, the Lie triple system is Abelian and we denote it (type I).
2.  $\dim \mathfrak{M}' = 1$ . Choosing the base  $e_1, e_2, e_3$  in  $\mathfrak{M}$  such that,  $\mathfrak{M}' = \langle e_1 \rangle$  and  $\mathfrak{M} = \mathfrak{M}' + \langle e_2, e_3 \rangle$ .

We will introduce in consideration the linear transformation  $A, B, C : \mathfrak{M} \longrightarrow \mathfrak{M}$ , define as:

$$A = (e_1, e_2, -) = \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = (e_2, e_3, -) = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C = (e_3, e_1, -) = \begin{pmatrix} x & -\alpha - c & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

And if a skew symmetric form defined as  $\Phi(-, -) : \mathfrak{M} \times \mathfrak{M} \longrightarrow \mathbb{R}$ , such that  $(x, y, e_1) = \Phi(x, y)e_1$ . The dimension of  $\mathfrak{M}$  is 3, that is why there exists  $z \in \mathfrak{M}$ ,  $z \neq 0$ , such that  $\Phi(-, z) = 0$ . The following cases are possible:

- b.I. The skew-symmetric form  $\Phi$  is non zero and  $z$  is parallel to  $e_1$  ( $z \parallel e_1$ ), then in the base  $e_1, e_2, e_3$  the skew-symmetric form  $\Phi$  has the corresponding matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta \\ 0 & -\delta & 0 \end{pmatrix},$$

where  $\delta \neq 0$ . Adjusting  $e_3$  to  $1 \setminus \delta e_3$ , then  $\Phi(e_2, e_3) = 1$ ,  $\Phi(e_3, e_2) = -1$ , so that  $\alpha = 1$ ,  $a = x = 0$  and



$$A = (e_1, e_2, -) = \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = (e_2, e_3, -) = \begin{pmatrix} 1 & \beta & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C = (e_3, e_1, -) = \begin{pmatrix} 0 & -1-c & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The verification of the defined relation of Lie triple system shows that, with accuracy to the choice of the vector basis  $e_2$  and  $e_3$ , it is possible to afford the following realization of the operators  $A$ ,  $B$ ,  $C$  as:

$$A = 0, B = (e_2, e_3, -) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C = (e_3, e_1, -) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(type VII)

- b.II. The skew-symmetric form  $\Phi$  is non zero and  $z$  is not parallel to  $e_1$ , let  $z = e_2$ , then

$$A = (e_1, e_2, -) = \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = (e_2, e_3, -) = \begin{pmatrix} 0 & \beta & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C = (e_3, e_1, -) = \begin{pmatrix} -1 & -c & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The verification of the defined relations of Lie triple system, show that the indicated case has no realization.

- b.III. The skew-symmetric form  $\Phi$  is trivial. By completing the vector  $e_1$  with the arbitrary choose vector  $e_2$  and  $e_3$  up to the base, it is possible to realize the operator  $A$ ,  $B$ , and  $C$ :

$$A = (e_1, e_2, -) = \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = (e_2, e_3, -) = \begin{pmatrix} 0 & \beta & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C = (e_3, e_1, -) = \begin{pmatrix} 0 & -c & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The verification of the defined relations of Lie triple system, show that by a suitable choice of basis vectors  $e_2, e_3$  the following realization of operators  $A, B, C$  is possible:

- Abelian Type (Type above)
- $A = C = 0$ ,

$$B = (e_2, e_3, -) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(Type II)

This Lie triple system, is obtained by a direct multiplication of a Lie triple system of dimension two  $\langle e_1, e_2 \rangle$ , by an Abelian one dimensional  $\langle e_3 \rangle$ .

– -

$$A = (e_1, e_2, -) = \begin{pmatrix} 0 & \pm 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$B=C=0$ . (Type III)

–

$$A = (e_1, e_2, -) = \begin{pmatrix} 0 & \pm 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

,  $B=0$ ,

$$C = (e_3, e_1, -) = \begin{pmatrix} 0 & -1 & \mp 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(Type IV)

3.  $\dim \mathfrak{M}' = 2$  in particular,  $\mathfrak{M}'$  is a subsystem of dimension two in  $\mathfrak{M}$ . one can consider (refer to §2.)  $\forall a, b, c \in \mathfrak{M}'$

$$(a, b, c) = \beta(a, c)b - \beta(b, c)a$$

where

$$\beta = \begin{pmatrix} \pm 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and  $\mathfrak{M}'$  is a two-dimensional Abelian ideal in  $\mathfrak{M}$ . In the first case the choice of the base  $\mathfrak{M} = \langle e_1, e_2, e_3 \rangle$  such that  $\mathfrak{M}' = \langle e_1, e_2 \rangle$ , the operations of the Lie triple system are reduced to:

$$A = (e_1, e_2, -) = \begin{pmatrix} 0 & \pm 1 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, B = (e_2, e_3, -) = \begin{pmatrix} \alpha & \gamma & \mu \\ \beta & \delta & \nu \\ 0 & 0 & 0 \end{pmatrix},$$

$$C = (e_3, e_1, -) = \begin{pmatrix} \kappa & -x - \alpha & \xi \\ \chi & -y - \beta & \beta \\ 0 & 0 & 0 \end{pmatrix}.$$

The verification of the defined relation of Lie triple system, leads to the contradiction of the condition that  $\dim \mathfrak{M}' = 2$ .

Let  $\mathfrak{M}' = \langle e_1, e_2 \rangle$  be a two-dimensional Abelian ideal and  $e_3$  the vector completing  $e_1, e_2$  up to the basis. Then:

$$A = (e_1, e_2, -) = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, B = (e_2, e_3, -) = \begin{pmatrix} \alpha & \gamma & \mu \\ \beta & \delta & \nu \\ 0 & 0 & 0 \end{pmatrix},$$

$$C = (e_3, e_1, -) = \begin{pmatrix} \kappa & -a - \alpha & \xi \\ \chi & -b - \beta & \beta \\ 0 & 0 & 0 \end{pmatrix}.$$

Deforming the vector  $e_1$  in the limit of the subspace  $\langle e_1, e_2 \rangle$ , the matrix  $A$  can be reduced to the form  $a = b = 0$  or  $a = 1, b = 0$ .

The verification of the defined relation of the Lie triple system, in the second case leads to the following realization of the operators  $A, B, C$ :

$$A = 0, B = (e_2, e_3, -) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C = (e_3, e_1, -) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

( type V)

$$A = 0, B = (e_2, e_3, -) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \pm 1 \\ 0 & 0 & 0 \end{pmatrix}, C = 0.$$

(type VI)

In conclusion to the conducted examination we have the following theorem:

**Theorem II.3.1.** Let  $\mathfrak{M} = \langle e_1, e_2, e_3 \rangle$  be a solvable Lie triple system of dimension 3,  $\mathfrak{G}$  its canonical enveloping Lie algebra(solvable), and let  $A, B, C : \mathfrak{M} \rightarrow \mathfrak{M}$  the linear transformations of the view:  $A = (e_1, e_2, -)$ ,  $A = (e_1, e_2, -)$ ,  $B = (e_2, e_3, -)$ ,  $C = (e_3, e_1, -)$ : with isomorphism accuracy, one can find the possibility of the following types:

- Type I.  $\mathfrak{M}$ - Abelian Lie triple system.
- Type II.

$$A = 0, C = 0, B = (e_2, e_3, -) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$  four-dimensional non-decomposable nilpotent Lie algebra with defined relations

$$[e_2, e_3] = e_4, [e_3, e_4] = -e_1$$

(this is  $g_{4,1}$  algebra in Mubarczyanov classification[37]).

- Type III.  $\mathfrak{M}$  is a direct product of a two-dimensional solvable Lie triple system  $\langle e_1, e_2 \rangle$ , and a one-dimensional Abelian  $\langle e_3 \rangle$  :

$$A = (e_1, e_2, -) = \begin{pmatrix} 0 & \pm 1 & 1 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, B = 0, C = 0$$

$\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$  four-dimensional solvable and decomposable Lie algebra, with defined relations:

$$[e_1, e_2] = e_4, [e_2, e_4] = \pm e_1$$

moreover  $\mathfrak{G} = \langle e_1, e_2, e_4 \rangle \oplus \langle e_3 \rangle$ , where  $\langle e_1, e_2, e_4 \rangle$ - three-dimensional solvable Lie (algebra  $g_{3,4 \setminus 5}$  in Mubarczyanov classification [37]).

- Type IV.

$$A = (e_1, e_2, -) = \begin{pmatrix} 0 & \pm 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = 0, C = (e_3, e_1, -) = \begin{pmatrix} 0 & -1 & \pm 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

,

$\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$  four-dimensional solvable and non-decomposable Lie algebra, with defined relations:

$$[e_1, e_2] = e_4, [e_2, e_4] = \pm e_1$$

$$[e_1, e_3] = \pm e_4, [e_3, e_4] = -e_1$$

(algebra  $g_{4,5 \setminus 6}$  in Mubarczyanov classification [37]).

- Type V

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \pm 1 \\ 0 & 0 & 0 \end{pmatrix}, A = C = 0$$

$\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$ - four-dimensional solvable non-decomposable Lie algebra with defined relations:

$$[e_2, e_3] = e_4, [e_2, e_4] = -e_1$$

$$[e_3, e_4] = \mp e_2$$

(algebra  $g_{8 \setminus 9}$  in Mubarakzyanov classification [37]).

- Type VI

$$A = 0, B = (e_2, e_3, -) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C = (e_3, e_1, -) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathfrak{G} = \langle e_1, e_2, e_3, e_4, e_5 \rangle$ - five-dimensional solvable non-decomposable Lie algebra, with defined relations:

$$[e_1, e_2] = e_4, [e_1, e_3] = -e_5$$

$$[e_3, e_4] = -e_1, [e_3, e_5] = -e_2$$

(as a result we obtain an extension of four-dimensional Abelian ideal  $\mathfrak{G} = \langle e_1, e_2, e_4, e_5 \rangle$  by means of  $\langle e_3 \rangle$ , algebra  $g_{4,13}$  in Mubarakzyanov classification [37]).

- Type VII.

$$A = 0, B = (e_2, e_3, -) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C = (e_3, e_1, -) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4, e_5 \rangle$ - five- dimensional solvable non-decomposable Lie algebra, with defined relations:

$$[e_2, e_3] = e_4, [e_1, e_3] = e_5$$

$$[e_1, e_4] = -e_1, [e_2, e_5] = -e_1, [e_4, e_5] = e_5$$

(algebra  $g_{4,11}$  in Mubarakzyanov classification [37]).

### 3.4 EXAMPLES OF 3-DIMENSIONAL BOL ALGEBRAS WITH SOLVABLE TRILINEAR OPERATIONS

- Example I. Solvable Lie algebras

Let  $\mathfrak{L}$ - be a 3-dimensional Lie algebra with basis  $e_1, e_2, e_3$  and  $C_{jk}^i$  ( $C_{jk}^i = -C_{kj}^i$ ) as its structural constants. The structural constant  $C_{jk}^i$  can be represented as:

$$C_{jk}^i = \varepsilon_{ijk} b^{li} + \delta_k^i a_k - \delta_j^i a_k$$

where  $\varepsilon_{ijk}$ - discriminant tensor,  $b^{li}$  symmetrical tensor such that  $(b^{li} = b^{il})$ ,  $\delta_k^i$  - Kronecker symbol and  $a_j$  covector, defined such that  $b^{ij} a_j = 0$ . Reducing  $b^{li}$  to the diagonal view and, by examination of the following 3-dimensional Lie algebras (called Bianchi classification [19]). Can be obtained the following non isomorphic Lie algebras):

- I.  $[e_i, e_j] = 0, i, j=1,2,3$
- II.  $[e_1, e_2] = 0, [e_1, e_3] = 0, [e_2, e_3] = e_1$
- III.  $[e_1, e_2] = 0, [e_1, e_3] = e_1, [e_2, e_3] = 0$
- IV.  $[e_1, e_2] = 0, [e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$
- V.  $[e_1, e_2] = 0, [e_1, e_3] = e_1, [e_2, e_3] = e_2$
- VI.  $[e_1, e_2] = 0, [e_1, e_3] = e_1, [e_2, e_3] = \lambda e_2, \lambda \neq 0, 1$
- VII.  $[e_1, e_2] = 0, [e_1, e_3] = e_2, [e_2, e_3] = -e_1 \mu + e_2 (\mu^2 < 4)$
- VIII.  $[e_1, e_2] = e_1, [e_1, e_3] = 2e_2, [e_2, e_3] = e_3$
- XI.  $[e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_2, e_3] = e_1$

Let's introduce in this examination the trilinear operation corresponding to the Lie triple systems:

$$(e_i, e_j, e_k) = [[e_i, e_j], e_k] = C_{ij}^t \cdot C_{tk}^s e_s$$

and matrix

$$A = (e_1, e_2, -), B = (e_2, e_3, -), C = (e_1, e_3, -).$$

Applying it to the distinguished Lie algebras above (I-IX) we obtain the following (with the omission of the zero relation):

- I-II.  $A = B = C = 0$  (Abelian case).
- III.  $(e_1, e_3, e_3) = e_1$  (type II in Ch.II §3).
- IV.  $(e_1, e_3, e_3) = e_1, (e_2, e_3, e_3) = e_2$  (type VI in Ch. II §3).
- V.  $(e_1, e_3, e_3) = e_1, (e_2, e_3, e_3) = e_2$  (subcase of type VI Ch. II §3).
- VI.  $(e_1, e_3, e_3) = e_1, (e_2, e_3, e_3) = \lambda^2 e_2, \lambda \neq 0, 1.$

- VII.  $(e_1, e_3, e_3) = -e_1 + \mu e_2$ ,  $(e_2, e_3, e_3) = -\mu e_1 + (1 - \mu^2)e_2$  ( $\mu^2 < 4$ ) (also allows imbedding in Type VI of classification in Ch. II §3).
- $(e_1, e_3, e_3) = -e_1$ ,  $(e_1, e_3, e_3) = 2e_3$ ,  $(e_1, e_2, e_3) = 2e_2$ ,  $(e_1, e_3, e_1) = -2e_1$ ,  $(e_2, e_3, e_1) = -2e_2$ ,  $(e_2, e_3, e_2) = -e_3$  (simple Lie triple system).
- IX.  $(e_1, e_2, e_2) = -e_1$ ,  $(e_1, e_3, e_1) = e_3$ ,  $(e_1, e_2, e_3) = e_2$ ,  $(e_1, e_3, e_3) = -e_1$ ,  $(e_2, e_3, e_2) = e_3$ ,  $(e, e_3, e_3) = -e_2$  (simple Lie triple system).

Choosing the solvable Lie triple system we obtain the following table:

	Defining relation of Lie algebra	Corresponding type of LTS
I	Abelian	Abelian
II	$[e_2, e_3] = e_1$	Abelian
III	$[e_1, e_3] = e_1$	$\langle e_1 \rangle \oplus$ <sup>1</sup> Type III
IV	$[e_2, e_3] = e_1 + e_2$ , $[e_1, e_3] = e_1$	$\langle e_3 \rangle \oplus \langle e_1, e_2 \rangle$ Type VI
V	$[e_2, e_3] = e_2$ , $[e_1, e_3] = e_1$	$\mathbb{R} \oplus$ <sup>2</sup> Type VI
VI	$[e_2, e_3] = \lambda e_2$ $\lambda \neq 0, 1$ , $[e_1, e_3] = e_1$	Type Vi
VII	$[e_2, e_3] = -e_1 + \mu e_2$ $\mu^2 < 4$ , $[e_1, e_3] = e_2$	Type VI

- Example 2. Classification of the right-alternative 3-dimensional algebras

A linear algebra  $V$  with the composition law  $\{x, y\}$  is called right-alternative if  $\forall x, y \in V$

$$\{y, \{x, x\}\} = \{\{y, x\}, x\}$$

In the work [29] it was shown that, over any field of characteristic zero with isomorphism accuracy there exist only five types of 3-dimensional right-alternative algebras. The indicated algebras are noted  $A$ ,  $B$ ,  $C$ ,  $E$ ,  $H$ .

Algebras  $A$ ,  $B$ ,  $C$ ,  $E$  and  $H$  are defined by the following relations:

- $A: \{e_2, e_3\} = e_2, \{e_3, e_2\} = e_1, \{e_3, e_3\} = e_3$
- $B: \{e_1, e_3\} = e_1, \{e_3, e_1\} = e_1, \{e_3, e_2\} = e_1 + e_2, \{e_3, e_3\} = e_3$
- $C: \{e_1, e_3\} = e_1, \{e_2, e_3\} = e_2, \{e_3, e_1\} = e_1 + e_2, \{e_3, e_3\} = e_3$
- $E: \{e_2, e_2\} = e_1, \{e_2, e_3\} = e_2, \{e_3, e_2\} = \beta e_1, \{e_3, e_3\} = e_3$
- $H: \{e_1, e_3\} = e_1, \{e_2, e_2\} = e_1, \{e_2, e_3\} = \gamma e_1, \{e_3, e_3\} = e_3, \{e_3, e_1\} = e_1, \{e_3, e_2\} = e_2$

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<sup>1</sup>2-dimensional solvable Lie triple system

<sup>2</sup>2-dimensional solvable Lie triple system

for any arbitrary  $\beta, \gamma$ .

According to [32], any right-alternative algebra is a Bol algebra under the operations:

$$\begin{aligned} x \cdot y &= \{x, y\} - \{y, x\} \\ < x, y, z > &= \{\{x, y\}, z\} - \{x, \{y, z\}\} \\ (x, y, z) &= z \cdot (x \cdot y) + 2 < z, x, y > . \end{aligned}$$

In the case of the algebra  $A$  we obtain

$$e_2 \cdot e_3 = e_2 - e_1, \quad (e_2, e_3, e_3) = -e_1 - e_2.$$

In the case of algebras  $B$  we obtain

$$e_2 \cdot e_3 = e_2 + e_1, \quad (e_2, e_3, e_3) = e_1 - e_2.$$

Here we see that the linear mapping with matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

realizes the isomorphism of Bol algebra  $A$  and  $B$ . The so obtained Bol algebra has Type  $III^+$  in chapter III §3 . classifications.

The composition law, corresponding to the local analytical Bol loop, can be follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \star \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + x_2 y_3 \\ x_2 + y_2 + x_3 y_2 \\ x_3 + y_3 + x_3 y_3 \end{pmatrix}$$

In the case of algebra  $C$

$$\begin{aligned} e_1 \cdot e_3 &= e_2, \quad (e_2, e_3, e_3) = -e_2 \\ e_2 \cdot e_3 &= -e_2, \quad (e_1, e_3, e_3) = e_2. \end{aligned}$$

In the base  $\tilde{e}_1 = e_1 + e_2, \tilde{e}_2 = e_2, \tilde{e}_3 = e_3$  the defined relations turn to

$$\begin{aligned} (\tilde{e}_1, \tilde{e}_3, \tilde{e}_3) &= 0, \quad \tilde{e}_1 \cdot \tilde{e}_3 = 0 \\ (\tilde{e}_1, \tilde{e}_3, \tilde{e}_3) &= -\tilde{e}_2, \quad \tilde{e}_1 \cdot \tilde{e}_3 = -\tilde{e}_2. \end{aligned}$$

Transforming the base again  $\bar{e}_1 = \tilde{e}_2, \bar{e}_2 = \tilde{e}_3, \bar{e}_3 = \tilde{e}_2$ , we obtain an isomorphism of the given Bol algebra with algebra type  $III^+$  ( $x = 1$ ), see Chapter III §3.



The composition law corresponding to the local analytic Bol loop, can be described as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \star \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + x_1 y_3 + x_3 y_1 \\ x_2 + y_2 + x_1 y_3 + x_3 y_2 \\ x_3 + y_3 + x_3 y_3 \end{pmatrix}$$

In the case of algebra  $E$

$$\begin{aligned} e_2 \cdot e_3 &= e_2 - \beta e_1, \quad (e_2, e_3, e_2) = -e_1 \\ (e_2, e_3, e_3) &= -e_2 - \beta e_1. \end{aligned}$$

After transforming the base

$$\bar{e}_1 = e_1; \bar{e}_2 = -e_2 - \beta e_1; \bar{e}_3 = e_3$$

we obtain

$$(\bar{e}_2, \bar{e}_3, \bar{e}_3) = -\bar{e}_2; (\bar{e}_2, \bar{e}_3, \bar{e}_2) = -\bar{e}_1; \bar{e}_2 \cdot \bar{e}_3 = \bar{e}_2 + 2\beta \bar{e}_1.$$

In the case of algebra  $H$

$$e_2 \cdot e_3 = -e_2 + \gamma e_1; (e_2, e_3, e_2) = 2e_1; (e_2, e_3, e_3) = -e_2 + \gamma e_1.$$

After the first transforming of the base

$$\bar{e}_1 = 2e_1; \bar{e}_2 = e_2; \bar{e}_3 = e_3,$$

we obtain

$$(\bar{e}_2, \bar{e}_3, \bar{e}_3) = \alpha \bar{e}_1 - \bar{e}_2;$$

$$(\bar{e}_2, \bar{e}_3, \bar{e}_2) = \bar{e}_1;$$

$$\bar{e}_2 \cdot \bar{e}_3 = -\bar{e}_2 + \alpha \bar{e}_1.$$

After the second transformation of the base

$$\hat{e}_1 = \bar{e}_1,$$

$$\hat{e}_2 = \alpha \bar{e}_1 - \bar{e}_2,$$

$$\hat{e}_3 = -\bar{e}_3, \text{ we obtain:}$$

$$(\hat{e}_2, \hat{e}_3, \hat{e}_3) = -\hat{e}_2,$$

$$(\hat{e}_2, \hat{e}_3, \hat{e}_2) = -\hat{e}_1,$$

$$\hat{e}_2 \cdot \hat{e}_3 = \hat{e}_2.$$

The linear map with matrix

$$\begin{pmatrix} -1 & 2\beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for any  $\beta$  realize the isomorphism of  $H$  and  $E$ . Then the trilinear composition law has Type V in Chapter III §3. classification. The composition law of local analytical Bol loop, can be described as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \star \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + x_2 y_2 + x_3 y_2 + x_2 y_3 \\ x_2 + y_2 + x_3 y_2 \\ x_3 + y_3 + x_3 y_3 \end{pmatrix}$$

• REMARK ABOUT THE CLASSIFICATION OF BOL 3-WEBS

In the work [55] the classification of six-dimensional Bol 3-Webs was examined. Let  $M^6$  be a smooth manifold given in the six-dimensional Bol 3-Webs  $W_6$ . Then three families of surfaces will define three perfectly integrable Pfaffian system of equations:  $w_1^i = 0, w_2^i = 0, w_1^i + w_2^i = 0$ , where  $i=1,2,3$ . The equations of structure of 3-Webs can be written as follows:

$$\begin{aligned} dw_1^i &= w_1^j \wedge \theta_j^i + \varepsilon_{jkl} a^{il} w_1^j \wedge w_1^k \\ dw_2^i &= w_2^j \wedge \theta_j^i - \varepsilon_{jkl} a^{il} w_2^j \wedge w_2^k \\ d\theta_j^i &= \theta_j^k \wedge \theta_k^i + \varepsilon_{klm} b^{im} w_1^k \wedge w_2^l \\ a^{ij} &= a^{ij} \theta_p^p + \frac{1}{2} (b_m^{ij} - b_p^{ip} \delta_m^i) (w_1^m - w_2^m) \end{aligned}$$

Moreover the tensor  $a^{ij}$ ,  $(i, j = 1, 2, 3)$  can be represented as  $a^{ij} = b^{ij} + c^{ij}$ , where  $b^{ij} = a^{(ij)}$ - symmetrical tensor,  $c^{ij} = a^{[ij]}$ - skew-symmetrical tensor. As the tensor  $c^{ij}$  is skew-symmetrical, then its rank can only be equal to zero or two.

Let the rang of  $c^{ij}$  be equal to two, then after a suitable transformation and normalizing its frame, the tensor  $a^{ij}$  can be reduced to one of the following types:

$$a^{ij} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

where  $\epsilon = \pm 1; \quad \epsilon_1 = \pm 1; 0$

$$a^{ij} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & a & 1 \\ 0 & -1 & \pm a \end{pmatrix}$$

where  $\epsilon = \pm 1, 0$  and for any  $a$ ,

$$a^{ij} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & \epsilon_1 \end{pmatrix}$$

where  $\epsilon = \pm 1, 0$   $\epsilon_1 = \pm 1, 0$ .

In this connection, only the last one verifies the condition of compatibility of the equations (1) [55].

$$b_k^{ik} = 2\varepsilon_{jkl}a^{il}a^{jk}$$

$$b_p^{ij}a^{pk} - b_p^{jk}a^{ip} - b_p^{ik}a^{pj} = b_p^{ip}a^{jk} - b_p^{pk}a^{ij}$$

which represents equations (1) ordinarily as

$$a_{jk}^i = \varepsilon_{jkl}a^{il}, \quad b_{kl}^i = \varepsilon_{klp}b_j^{ip}$$

where  $\varepsilon_{jkl}$ ,  $\varepsilon_{klp}$  are discriminant tensors. As shown in [55] the skew-symmetric tensor  $a^{ij}$

$$a^{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

is the necessary and sufficient condition for six-dimensional isocline Bol Web.

In other words, if the tensor  $a^{ij}$  (six-dimensional Bol three-Web) is symmetric and the the rank of  $a^{ij}$  equals zero, the Web in this case is parallelizable.

The trilinear operations of Bol algebras corresponding to the given three-Webs, with tensor  $a^{ij}$  of type above, define in [55], prove to be simple and splittable (in the sense of the terminology of Chapter I §5.).

## 4 CHAPTER III

### CLASSIFICATION OF 3-DIMENSIONAL BOL ALGEBRAS AND CALCULUS OF CORRESPONDING BOL 3-WEBS

#### 4.1 BOL ALGEBRAS WITH TRIVIAL TRILINEAR OPERATIONS OF TYPE I

Let  $\mathfrak{M}$  be a Bol algebra [46] with a trivial trilinear operation. The structure of  $\mathfrak{M}$  will be reduced, to the representation of an anti-commutative bilinear multiplication  $(\cdot)\mathfrak{M} \times \mathfrak{M} \longrightarrow \mathfrak{M}$ , such that:

$$(X \cdot Y) \cdot (Z \cdot U) = 0$$

$\forall X, Y, Z, U \in \mathfrak{M}$

Possible case:

1.  $\xi \cdot \eta = 0$  (abelian case)
2.  $\xi \cdot \eta \zeta = 0$  the algebra  $\mathfrak{M}$  - not abelian (but anticommutative 2-nilpotent algebra),
3.  $(\xi \eta) \cdot (\zeta \kappa) = 0$ , but the algebra  $\mathfrak{M}(\cdot)$  not 2-nilpotent algebra).

(In particular,  $\mathfrak{M}(\cdot)$  can be anticommutative 3-nilpotent algebra) hence:

$$(\xi \eta \cdot \zeta) \cdot \kappa = 0$$

$\forall \xi, \eta, \zeta, \kappa \in \mathfrak{M}$

In this investigation we are considering the question **how long will (with accuracy to the isomorphism) exist the operation viewed in (1)-(3) in the case of 3-dimensional algebra  $\mathfrak{M}$ .**

We will consider (with accuracy to the isomorphism) the uniqueness of the abelian algebra, denoted Type III.1 and hence investigate case 2.

Case 2. Let's denote the subspace  $\mathfrak{M} \cdot \mathfrak{M}$  in  $\mathfrak{M}$  by  $V$  then  $V \cdot \mathfrak{M} = 0$  and  $V \neq \mathfrak{M}$ ,  $V \neq 0$ . The following variants can be obtained:

- 2.a  $\dim V = 1$ , then consider that,  $\mathfrak{M} = \langle e_1, e_2, e_3 \rangle$ ,  $V = \langle e_1 \rangle$ ,

$$e_1 \cdot e_2 = e_1 \cdot e_3 = 0$$

$$e_2 \cdot e_3 = \alpha e_1, \alpha \neq 0$$

by adjusting the base  $e_1, e_2, e_3$ , we make

$$e_2 \cdot e_3 = e_1,$$

the so obtained algebra will be denoted by algebra Type III.2.

- 2.b.  $\dim V = 2$ , then  $\mathfrak{M} = \langle e_1, e_2, e_3 \rangle$ ,  $V = \langle e_1, e_2 \rangle$ , and  $e_1 \cdot e_2 = e_1 \cdot e_3 = 0$ ,  $e_2 \cdot e_3 = 0$ , (1)

The relation in (1) are in contradiction with the condition  $\mathfrak{M} \cdot \mathfrak{M} = V$ . Hence there exist with isomorphism accuracy only one Bol algebra of case 2.

Now we past to the examination of Bol algebra case 3.

Let us suppose that  $\mathfrak{M} \cdot \mathfrak{M} = V \subseteq \mathfrak{M}$ , thus:

$$V \cdot V = \{O\} \text{ but } V \cdot \mathfrak{M} \neq \{O\} \text{ } V \neq \mathfrak{M}, V \neq \{O\}.$$

- 3.a Let  $\dim V = 1$  and  $V = \langle e_1 \rangle$ , where  $e_1, e_2, e_3$ - are basis vectors in  $\mathfrak{M}$ , then:

$$\begin{aligned} e_1 \cdot e_2 &= \alpha e_1, e_1 \cdot e_3 = \beta e_1, \\ e_2 \cdot e_3 &= \gamma e_1, \alpha^2 + \beta^2 \neq 0. \end{aligned}$$

Without limiting our selves, one can consider  $\beta \neq 0$ , by changing the base  $e_1, e_2, e_3$  the defined relation of anti-commutative algebra  $\mathfrak{M}$  can be reduced to:

$$\begin{aligned} e_1 \cdot e_2 &= 0, e_1 \cdot e_3 = e_1, \\ e_2 \cdot e_3 &= \varepsilon e_1, \text{ where } \varepsilon = 1, 0. \end{aligned}$$

Let us denote the defined algebra through  $\mathfrak{M}_0$  and  $\mathfrak{M}_1$  correspondently, and examine its isomorphism. We will note:

$$\mathfrak{M}_0 = \{O\} \quad \mathfrak{M}_1 = \{e_1\},$$

that means their centers are different, hence algebras  $\mathfrak{M}_0$  and  $\mathfrak{M}_1$  are not isomorphic. We will denote them by, Type III.3 and Type III.4 correspondently.

- 3.b. Let  $\dim V = 2$ , and let's consider  $e_1, e_2, e_3$  the basis in  $\mathfrak{M}$  such that  $V = \langle e_1, e_2 \rangle$ , thus:

$$e_1 \cdot e_2 = 0$$

and

$$e_1 \cdot e_3, e_2 \cdot e_3 \in V$$

Deforming  $e_3$  to  $\tilde{e}_3 = te_3 + ve_2 + ue_1$ , where  $t \neq 0$   $u, v \in \mathbb{R}$  we obtain

$$e_1 \cdot \tilde{e}_3 = e_1 \cdot (te_3 + ve_2 + ue_1) = te_1 \cdot e_3,$$

$$e_2 \cdot \tilde{e}_3 = e_2 \cdot (te_3 + ve_2 + ue_1) = te_2 \cdot e_3,$$

we can consider that, the transformation

$$\Phi = \text{ade}_3| : V \longrightarrow V$$

$$\Phi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is defined with accuracy to the multiplicative scalar.

The choice of the basis  $e_1, e_2$  in  $V$  and vector  $e_3$  we can reduce  $\phi$  to one of the following:

$$\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

where  $\alpha \neq 0$  and  $\forall \mu \in \mathbb{R}$

Respectively

$$e_1 \cdot e_3 = e_1,$$

$$e_2 \cdot e_3 = \mu e_2$$

or

$$e_1 \cdot e_3 = e_1 + \alpha e_2$$

$$e_2 \cdot e_3 = e_2$$

Let us examine the first case: In that case  $\mu \neq 0$  (otherwise we will obtain the contradiction with  $\dim V = 2$ ), by adjusting  $e_2$  to  $\tilde{e}_2 = \mu e_2$  we obtain:

$$e_1 \cdot e_3 = e_1,$$

$$e_2 \cdot e_3 = e_2$$

(algebra Type III.5)

In the second case the change of  $(1 \setminus a)e_1 \longrightarrow e_1$  reduces the defined relations of algebra  $\mathfrak{M}$  to:

$$e_1 \cdot e_3 = e_1 + e_2$$

$$e_2 \cdot e_3 = e_2$$

(algebra Type III.6). The algebra  $\mathfrak{M}$  of Type III.5 and Type III.6 are obtained by the extension 1-dimensional Abelian algebra by means of 2-dimensional Abelian algebra  $V = \mathfrak{M} \cdot \mathfrak{M} = \langle e_1, e_2 \rangle$ , hence for  $a \in \mathfrak{M} \setminus V$  the structure of the operator  $\text{ad}|_V$  is:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

$\alpha \neq 0$  (algebras Type III.5)

$$\begin{pmatrix} \alpha & 0 \\ \alpha & \alpha \end{pmatrix}$$

$\alpha \neq 0$  (algebras Type III.6)

that's why algebra of Type III.5 and Type III.6 are not isomorphic.

In result we obtain the following theorem:

**Theorem III.1** There exists with isomorphism accuracy 6 Bol algebras, with a trivial trilinear operation defined as follows:

- III.1 trivial bilinear operation
- III.2  $e_2 \cdot e_3 = e_1$
- III.3  $e_1 \cdot e_3 = e_1$
- III.4  $e_2 \cdot e_3 = e_1, e_1 \cdot e_3 = e_1$
- III.5  $e_2 \cdot e_3 = e_2, e_1 \cdot e_3 = e_1$
- III.6  $e_2 \cdot e_3 = e_2, e_1 \cdot e_3 = e_1 + e_2$

We will note the enveloping Lie algebras of Bol algebras III.3 and Bol algebras III.4 are 4-dimensional, but these Bol algebras are not isotopics by their structure. The enveloping Lie algebras of Bol algebras III.5 and Bol algebras III.6 are identical, but these Bol algebras are not isotopic by definition.

Below we will give the description of 3-webs, corresponding to the defined Bol algebras.

- Type III.1. Bol algebra  $\mathfrak{B}$  with trivial trilinear and bilinear operation. Here we obtain grouped 3-Webs (Abelian group  $\langle \mathbb{R}^3, +, 0 \rangle$ ).
- Type III.2. Bol algebras  $\mathfrak{B}$  with bilinear anti-commutative operation of view:

$$e_2 \cdot e_3 = e_1.$$

We also obtain here a grouped 3-Webs (global 3-Web) corresponding to the Lie group, which is isomorphic to the upper triangular unipotent matrix, which means matrix of form:

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

- Type III.3 Bol algebras  $\mathfrak{B}$  with bilinear operation of View:

$$e_1 \cdot e_3 = e_1.$$

and trivial trilinear operation, has a 4-dimensional canonical enveloping Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$ ,

$$\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}, \mathfrak{B} = \langle e_1, e_2, e_3 \rangle, \mathfrak{h} = \langle e_4 - e_4 \rangle$$

with a composition law having the following constant of structure

$$[e_1, e_3] = e_4$$

the composition law ( $\triangle$ ) corresponding to the Lie group  $G$  is defined as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \triangle \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 + \frac{x_1 y_3 - y_1 x_3}{2} \end{bmatrix}$$

Moreover the subgroup  $H = \exp \mathfrak{h}$ , realized as a collection of elements:

$$\{\exp t(e_4 - e_1)\}_{t \in \mathbb{R}} = \{(\exp t e_4)(\exp t e_1)^{-1}\}_{t \in \mathbb{R}} = \{-t, 0, 0, t\}_{t \in \mathbb{R}}.$$

The collection of elements

$$B = \exp \mathfrak{B} = \{\exp(t e_1 + u e_2) \cdot \exp v e_3\}_{t, u, v \in \mathbb{R}} = \{t, u, v, t\}_{t, u, v \in \mathbb{R}}$$

form a local section of space left coset  $G \bmod H$ .

The subgroup  $H$  is defined as:

$$H = \exp \mathfrak{h} = \{\exp \alpha(e_4 - e_1)\}_{\alpha \in \mathbb{R}} = \{0, 0, -\alpha, \alpha\}_{\alpha \in \mathbb{R}},$$

$$B = \{t, u, v, 0\}.$$

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$  such that  $(X_3 < -2)$  can be uniquely represented in the form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} \frac{2x_1 + 2x_4 - x_1 x_3}{2 - x_3} \\ x_2 \\ x_3 \\ 0 \end{bmatrix} \triangle \begin{bmatrix} -\frac{2x_4}{2 - x_3} \\ 0 \\ 0 \\ \frac{2x_4}{2 - x_3} \end{bmatrix}$$



$$= \prod_B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \triangle \begin{bmatrix} -\frac{2x_4}{2-x_3} \\ 0 \\ 0 \\ \frac{2x_4}{2-x_3} \end{bmatrix}.$$

The composition law( $\star$ ) of local analytic Bol loop  $B(\star)$  is defined as:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \triangle \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} t + t' + \frac{tv' - vt'}{2 - (v + v')} \\ u + u' \\ v + v' \end{bmatrix}. \end{aligned}$$

The corresponding local Bol 3-Webs can be realized in the neighborhood of the point  $(O, O)$  in  $\mathbb{R}^6 = \{(\bar{X}, \bar{Y}), \bar{X}, \bar{Y} \in \mathbb{R}^3\}$  as a space form by:

$$\begin{aligned} \bar{X} &= \text{const} \\ \bar{Y} &= \text{const} \\ \bar{X} \star \bar{Y} &= \text{const}. \end{aligned}$$

- Type III.4 Bol algebra  $\mathfrak{B}$  with bilinear operation of view:

$$e_1 \cdot e_3 = e_1, e_2 \cdot e_3 = e_1$$

an trivial trilinear operation, has a 4-dimensional canonical enveloping Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$ ,  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}$ ,  $\mathfrak{B} = \langle e_1, e_2, e_3 \rangle$ ,  $\mathfrak{h} = \langle e_4 - e_1 \rangle$

With composition law having the following structural equations

$$[e_1, e_3] = e_4, [e_2, e_3] = e_4.$$

The composition law ( $\triangle$ ), corresponding to the Lie group  $G$  is defined as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \triangle \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 + \frac{(x_1 + x_2)y_3 - (y_1 + y_2)x_3}{2} \end{bmatrix}$$

Moreover the subgroup  $H = \exp \mathfrak{h}$ , is realized as a collection of elements:

$$\{\exp t(e_4 - e_1)\}_{t \in \mathbb{R}} = \{(\exp te_4) \cdot (\exp te_1)^{-1}\}_{t \in \mathbb{R}} = \{-t, 0, 0, t\}_{t \in \mathbb{R}}.$$

The collection

$$B = \exp \mathfrak{B} = \{\exp(te_1 + ue_2) \cdot \exp ve_3\}_{t,u,v \in \mathbb{R}} = \{t, u, v, 0\}_{t,u,v \in \mathbb{R}}$$

form a local section of space left coset  $G \bmod H$ .

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$  such that  $(x_3 > -2)$ , can be uniquely represented in the form:

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{bmatrix} \frac{2x_1+2x_4+x_1x_3}{2+x_3} \\ x_2 \\ x_3 \\ 0 \end{bmatrix} \triangle \begin{bmatrix} -\frac{2x_4}{2+x_3} \\ 0 \\ 0 \\ \frac{2x_4}{2+x_3} \end{bmatrix} \\ &= \prod_B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \triangle \begin{bmatrix} -\frac{2x_4}{2+x_3} \\ 0 \\ 0 \\ \frac{2x_4}{2+x_3} \end{bmatrix}. \end{aligned}$$

The composition law(  $\star$  ), of local analytic Bol loop  $B(\star)$  is defined as:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \prod_B \left( \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \triangle \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} t + t' + \frac{(t+u)v' - v(t'+u')}{2+(v+v')} \\ u + u' \\ v + v' \end{bmatrix}. \\ &= \begin{bmatrix} \frac{2t+2t'+tv+tv'+t'v'+t'v+uv'-vu'}{2+(v+v')} \\ u + u' \\ v + v' \end{bmatrix}. \end{aligned}$$

The corresponding local Bol 3-Web, can be realized in the neighborhood of the point  $(O, O)$  in  $\mathbb{R}^6 = \{(\bar{X}, \bar{Y}), \bar{X}, \bar{Y} \in \mathbb{R}^3\}$  as a space of second order.

- Type III.5 Bol algebra  $\mathfrak{B}$ , with bilinear operation of view:

$$e_1 \cdot e_3 = e_1, \quad e_2 \cdot e_3 = e_2$$

an trivial trilinear operation, has a 5-dimensional canonical enveloping Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4, e_5 \rangle$ ,  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}$ ,  $\mathfrak{B} = \langle e_1, e_2, e_3 \rangle$ ,  $\mathfrak{h} = \langle e_4 - e_1, e_5 - e_2 \rangle$

With composition law having the following structural equations

$$[e_1, e_3] = e_4, \quad [e_2, e_3] = e_5.$$

The composition law ( $\Delta$ ) corresponding to the Lie group  $G$  is defined as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \Delta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 + \frac{x_1 y_3 - y_1 x_3}{2} \\ x_5 + y_5 + \frac{x_2 y_3 - x_3 y_2}{2} \end{bmatrix}$$

Moreover the subgroup  $H = \exp \mathfrak{h}$ , can be realized as a collection of elements:

$$\{\exp t(e_4 - e_1), \exp p(e_5 - e_2)\}_{t,p \in \mathbb{R}} = \{-t, -p, 0, t, p\}_{t,p \in \mathbb{R}}.$$

The collection

$$B = \exp \mathfrak{B} = \{\exp(te_1 + ue_2) \cdot \exp ve_3\}_{t,u,v \in \mathbb{R}} = \{t, u, v, 0, 0\}_{t,u,v \in \mathbb{R}}$$

form a local section of space left coset  $G \bmod H$ .

Any element  $(x_1, x_2, x_3, x_4, x_5)$  from  $G$  such that  $(x_3 > -2)$ , can be uniquely represented in the form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{bmatrix} \frac{2x_1 + 2x_4 + x_1 x_3}{2 + x_3} \\ \frac{2x_2 + 2x_5 + x_2 x_3}{2 + x_3} \\ x_3 \\ 0 \\ 0 \end{bmatrix} \Delta \begin{bmatrix} -\frac{2x_4}{2 + x_3} \\ -\frac{2x_5}{2 + x_3} \\ 0 \\ \frac{2x_4}{2 + x_3} \\ \frac{2x_5}{2 + x_3} \end{bmatrix}$$

$$= \prod_B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \triangleq \begin{bmatrix} -\frac{2x_4}{2+x_3} \\ -\frac{2x_5}{2+x_3} \\ 0 \\ \frac{2x_4}{2+x_3} \\ \frac{2x_5}{2+x_3} \end{bmatrix}.$$

The composition law  $(\star)$ , of local analytic Bol loop  $B(\star)$  is defined as:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} t + t' + \frac{tv' - vt'}{2+(v+v')} \\ u + u' + \frac{uv' - vu'}{2+(v+v')} \\ v + v' \end{bmatrix}. \\ &= \begin{bmatrix} \frac{2t+2t'+tv+2tv'+t'v'}{2+(v+v')} \\ \frac{2u+2u'+uv+2uv'+u'v'}{2+(v+v')} \\ v + v' \end{bmatrix}. \end{aligned}$$

The corresponding local Bol 3-Web, can be realized in the neighborhood of the point  $(O, O)$  in  $\mathbb{R}^6 = \{(\bar{X}, \bar{Y}), \bar{X}, \bar{Y} \in \mathbb{R}^3\}$  as a space of second order.

- Type III.6 Bol algebra  $\mathfrak{B}$  with bilinear operation of view:

$$e_1 \cdot e_3 = e_1 + e_2, \quad e_2 \cdot e_3 = e_2$$

an trivial trilinear operation, has a 5-dimensional canonical enveloping Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4, e_5 \rangle$ ,  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}$ ,  $\mathfrak{B} = \langle e_1, e_2, e_3 \rangle$ ,  $\mathfrak{h} = \langle e_4 - e_1 - e_2, e_5 - e_2 \rangle$ .

With composition law having the following structural equations

$$[e_1, e_3] = e_4, \quad [e_2, e_3] = e_5.$$

The composition law  $(\triangle)$ , corresponding to the Lie group  $G$  is defined as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \triangleq \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 + \frac{x_1 y_3 - y_1 x_3}{2} \\ x_5 + y_5 + \frac{x_2 y_3 - x_3 y_2}{2} \end{bmatrix}.$$

Moreover the subgroup  $H = \exp \mathfrak{h}$ , can be realized as a collection of elements:

$$\{\exp t(e_4 - e_1 - e_2), \exp p(e_5 - e_2)\}_{t,p \in \mathbb{R}} = \{-t, -t - p, 0, t, p\}_{t,p \in \mathbb{R}}.$$

The collection

$$B = \exp \mathfrak{B} = \{\exp(te_1 + ue_2) \cdot \exp ve_3\}_{t,u,v \in \mathbb{R}} = \{t, u, v, 0, 0\}_{t,u,v \in \mathbb{R}}$$

form a local section of space left coset  $G \bmod H$ .

Any element  $(x_1, x_2, x_3, x_4, x_5)$  from  $G$  such that  $(x_3 > -2)$ , can be uniquely represented in the form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{bmatrix} \frac{2x_1+2x_4+x_1x_3}{2+x_3} \\ \frac{x_2(2+x_3)^2+2x_4(2+x_3)+4x_5+2x_3x_5-2x_3x_4}{(2+x_3)^2} \\ x_3 \\ 0 \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} -\frac{2x_4}{2+x_3} \\ -\frac{4x_4+4x_5+2x_3x_5}{(2+x_3)^2} \\ 0 \\ \frac{2x_4}{2+x_3} \\ \frac{4x_5+2x_3x_5-2x_3x_4}{(2+x_3)^2} \end{bmatrix}$$

$$= \prod_B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \triangleq \begin{bmatrix} -\frac{2x_4}{2+x_3} \\ -\frac{4x_4+4x_5+2x_3x_5}{(2+x_3)^2} \\ 0 \\ \frac{2x_4}{2+x_3} \\ \frac{4x_5+2x_3x_5-2x_3x_4}{(2+x_3)^2} \end{bmatrix}.$$

The composition law  $(\star)$ , of local analytic Bol loop  $B(\star)$  is defined as:

$$\begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} = \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} t + t' + \frac{tv' - vt'}{2+(v+v')} \\ u + u' + \frac{tv' - vt'}{2+(v+v')} + \frac{uv' - vu'}{2+(v+v')} - \frac{(v+v')(tv' - vt')}{(2+(v+v'))^2} \\ v + v' \end{bmatrix}.$$

The corresponding local Bol 3-Web can be realized in the neighborhood of the point  $(O, O)$  in  $\mathbb{R}^6 = \{(\bar{X}, \bar{Y}), \bar{X}, \bar{Y} \in \mathbb{R}^3\}$  as a space of third order.

## 4.2 BOL ALGEBRAS WITH TRILINEAR OPERATION OF TYPE II

We will base our investigation of 3-dimensional Bol algebras on the examination of their canonical enveloping Lie algebras. In what follows, we consider Bol algebras of dimension 3, from their construction see [34, 46]; it follows that the dimension of their canonical enveloping Lie algebras can not be more than 6. Below we limit ourselves to the classification of Bol algebras (and their corresponding 3-Webs) with canonical enveloping Lie algebras of dimension  $\leq 5$ .

Let  $\mathfrak{B}$  be a 3-dimensional Bol algebras with a trilinear operation Type II in [11], and  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}$  - its canonical enveloping Lie algebra according to [11], we note that the situation  $\dim \mathfrak{G} = 3$  is impossible that means the case of Web-group is excluded.

Let us examine the case  $\dim \mathfrak{G} = 4$ , the structural constants of Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$ ,  $\mathfrak{B} = \langle e_1, e_2, e_3 \rangle$  are defined as follows:

$$[e_2, e_3] = e_4, \quad [e_3, e_4] = -e_1$$

here  $\mathfrak{G} = \mathfrak{B} \dot{+} [\mathfrak{B}, \mathfrak{B}]$ ,  $[\mathfrak{B}, \mathfrak{B}] = \langle e_4 \rangle$ .

Introducing the subalgebra

$$\mathfrak{h}_{\alpha, \beta, \gamma} = \langle e_4 + \alpha e_1 + \beta e_2 + \gamma e_3 \rangle$$

were  $\alpha, \beta, \gamma \in \mathbb{R}$ .

We are getting a collection of Bol algebras with the structural constants defined as follows:

$$e_2 \cdot e_3 = -\alpha e_1 - \beta e_2 - \gamma e_3, \quad (e_2, e_3, e_3) = e_1.$$

Below we give an isomorphical and an isotopical classification of this collection.

The group of automorphism  $F$  of Lie triple system  $\mathfrak{B}$  is defined as follows:

\*\*\*\*

$$ad(\xi) = \begin{pmatrix} 0 & 0 & 0 & -z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -z & y & 0 \end{pmatrix}$$

$$Ad(\xi) = \begin{pmatrix} 1 & \frac{z^2}{2} & -\frac{zy}{2} & -z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -z & y & 1 \end{pmatrix}$$

and transformations  $Ad\xi$ ,  $\xi \in \mathfrak{B}$  allow us to combine only case 4 from (3)

**Theorem** Any Bol algebra of dimension 3, with the trilinear operation of Type II, and the canonical enveloping Lie algebra of dimension 4 is isotopic to one of the algebras 2-4.

Below we will give the description of three-Webs corresponding to the distinguished Bol algebras.

The composition law of local Lie group  $G = \langle \mathbb{R}^4, \star, O \rangle$ , tangent to the Lie algebra  $\mathfrak{G}$  is defined as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \triangle \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + \frac{x_4 y_3 - x_3 y_4}{2} + \frac{x_3^2 y_2 - x_3 x_2 y_3}{12} + \frac{x_2 y_3^2 - x_3 y_2 y_3}{12} \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 + \frac{x_2 y_3 - y_2 x_3}{2} \end{bmatrix}.$$

Case 2. Bol algebra with bilinear and trilinear operation is defined as:

$$e_2 \cdot e_3 = -e_3, (e_2, e_3, e_3) = e_1$$

The subgroup  $H$  is defined as

$$H = \exp \mathfrak{h} = \{\exp \alpha(e_4 - e_3)\}_{\alpha \in \mathbb{R}}.$$

The subgroup  $H$  is defined as

$$H = \exp \mathfrak{h} = \{\exp \alpha(e_4 - e_3)\}_{\alpha \in \mathbb{R}} = \{0, 0, \alpha, \alpha\},$$

$$B = \{t, u, v, 0\}.$$

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$  such that  $(x_2 < 2)$ , can be uniquely represented in the form:

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{bmatrix} T \\ x_2 \\ \frac{2x_3 + 2x_4 - x_2 x_3}{2 - x_2} \\ 0 \end{bmatrix} \triangle \begin{bmatrix} 0 \\ 0 \\ -\frac{2x_4}{2 - x_2} \\ \frac{2x_4}{2 + x_2} \end{bmatrix} \\ &= \prod_B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \triangle \begin{bmatrix} 0 \\ 0 \\ -\frac{2x_4}{2 - x_2} \\ \frac{2x_4}{2 + x_2} \end{bmatrix}. \end{aligned}$$

Where

$$T = \frac{12x_4^2 - 4x_2(x_4)^2 + (x_2)^2 x_4 x_3 - 8x_2 x_3 x_4 + 12x_4 x_3 + 6x_1(2 - x_2)^2}{(6(2 - x_2)^2)}.$$

The composition law  $(\star)$ , of local analytic Bol loop  $B(\star)$  is defined as:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \triangle \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{L}{6(2-(u+u'))^2} \\ u + u' \\ v + v' + \frac{uv' - vu'}{2-u-u'} \end{bmatrix}. \end{aligned}$$

Where

$$\begin{aligned} L = & 24t + 24t' - 24tu - 24t'u - 24tu' - 24t'u' + 12tuu' + 12t'uu' \\ & + 6tu^2 + 6t'u^2 + 6t(u')^2 + 6t'(u')^2 - 2uvu' - 2uu'v' \\ & + 6uvv' - 6vu'u' - 6uu'v'v + 2u^2vu' + 2u^2u'v' - 4u^2vv' \\ & + 2u(v')^2 - 4v^2u' + 2uv^2u' + 2uv(u')^2 - 6uu'(v')^2 \\ & + 6u(v')^2 + 2u(u')^2v' + 4v(u')^2v' - 3u^2(v')^2 \\ & + 5v^2(u')^2 - u^2v(u')^2 + u^2u'(v')^2 \\ & - u^2(u')^2v' + u(u')^2(v')^2 - v^2(u')^3 + \frac{5u^2vu'v'}{2} - uv^2(u')^2 \\ & + \frac{3uvv'(u'^2)}{2} - \frac{u^3vu'}{2} - \frac{u'v'u^3}{2} - \frac{uv(u')^3}{2} \\ & - \frac{u(u')^3}{2} - \frac{vv'(u')^3}{2} + \frac{vv'(u)^3}{2}. \end{aligned}$$

The corresponding local 3-Webs can be realized as a 6-order space.

Bol algebra in case 3 with bilinear and trilinear operation is defined as:

$$e_2 \cdot e_3 = -e_2, (e_2, e_3, e_3) = e_1$$

has a 4-dimensional canonical enveloping Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$

$$\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}, \mathfrak{B} = \langle e_1, e_2, e_3 \rangle, \mathfrak{h} = \langle e_4 + e_2 \rangle$$

with the composition law indicated above the subgroup

$$H = \exp \mathfrak{h} = \{ \exp \alpha(e_4 + e_2) \}_{\alpha \in \mathbb{R}} = \{0, \alpha, 0, \alpha\}$$

$$B = \{t, u, v, 0\}.$$

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$  such that  $(x_2 < 2)$ , can be uniquely represented in the form:



$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{bmatrix} \frac{-6x_1x_3+6x_4x_3-x_4x_3^2+12x_1}{6(2-x_3)} \\ \frac{2x_2-2x_4-x_2x_3}{2-x_3} \\ x_3 \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} 0 \\ \frac{2x_4}{2-x_3} \\ 0 \\ \frac{2x_4}{2-x_3} \end{bmatrix} \\
&= \prod_B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \triangleq \begin{bmatrix} 0 \\ \frac{2x_4}{2-x_3} \\ 0 \\ \frac{2x_4}{2-x_3} \end{bmatrix}.
\end{aligned}$$

The composition law( $\star$ ), of local analytic Bol loop  $B(\star)$  is defined as:

$$\begin{aligned}
\begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} t + t' + \frac{B''}{6(2-v-v')} \\ u + u' + \frac{u'v-v'u}{2-v-v'} \\ v + v' \end{bmatrix}.
\end{aligned}$$

Where

$$B'' = v^2u' - uvv' + u(v)^2 - vu'v' - u'v^3 + uv'v^2 + 3uvv' - 3u'v^2 - 3vu'v' + uvv' - u'v'v^2.$$

The corresponding local 3-Webs can be realized as a 5-order space.

Bol algebra in case 4 with bilinear and trilinear operation is defined as:

$$e_2 \cdot e_3 = -e_1, (e_2, e_3, e_3) = e_1,$$

has a 4-dimensional canonical enveloping Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$ ,

$$\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}, \mathfrak{B} = \langle e_1, e_2, e_3 \rangle, \mathfrak{h} = \langle e_4 - e_1 \rangle$$

with the composition law indicated above. The subgroup

$$H = \exp \mathfrak{h} = \{ \exp \alpha(e_4 - e_1) \}_{\alpha \in \mathbb{R}} = \{ -\alpha, 0, 0, \alpha \}$$

$$B = \{ t, u, v, 0 \}.$$

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , can be uniquely represented in the form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} x_1 + x_4 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} -x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix}$$

$$= \prod_B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \triangle \begin{bmatrix} 0 \\ \frac{2x_4}{2-x_3} \\ 0 \\ \frac{2x_4}{2-x_3} \end{bmatrix}.$$

The composition law  $(\star)$ , of local analytic Bol loop  $B(\star)$  is defined as:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \triangle \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} t + t' + \frac{uv' - vu'}{2} + \frac{v^2 u' - uvv'}{12} + \frac{u(v')^2 - vu'v'}{12} \\ u + u' \\ v + v' \end{bmatrix}. \end{aligned}$$

The corresponding 3-web is global and can be realized as a 3-order space.

**Remark:** case 1 has not been investigated, because it is combined with case 4 see proposition 2.

We pass to the classification of Bol algebras of dimension 3, with trilinear operation of type II with 5-dimensional canonical enveloping Lie algebra.

The structural constants of Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4, e_5 \rangle$   $\mathfrak{G} = \mathfrak{B} \dot{+} V$ , must satisfy the relations:  $\mathfrak{B} = \langle e_1, e_2, e_3 \rangle$ ,  $[\mathfrak{B}, \mathfrak{B}] \subset V = \langle e_4, e_5 \rangle$ ,  $[V, \mathfrak{B}] \subset \mathfrak{B}$ ,  $[e_5, \mathfrak{B}] = 0$

$$[e_2, e_3] = e_4 + re_5,$$

$$[e_3, e_1] = se_5,$$

$$[e_4, e_3] = e_1,$$

(trivial relations have been omitted) in case where  $s = 0$ , we have  $\langle \mathfrak{B} \neq \mathfrak{G}$ , which means Lie algebras  $\mathfrak{G}$  is not a canonical enveloping. That is why by adjusting, if necessary vector  $e_5$  we have  $s = 1$  (for any  $r$ ).

Here we will have two families of subalgebras  $\mathfrak{h}$  in  $\mathfrak{G}$ , such that  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}$ .

$$\mathfrak{h} = \langle e_1 + \alpha e_2 + \beta e_3 + \gamma e_4, e_5 \rangle, \forall \alpha, \beta, \gamma \in \mathbb{R}$$

$$\mathfrak{h}_{\alpha, \beta, \bar{\alpha}, \bar{\beta}} = \langle e_4 + \alpha e_1 + \beta e_2, e_5 + \bar{\alpha} e_1 + \bar{\beta} e_2 \rangle, \forall \alpha, \beta, \bar{\alpha}, \bar{\beta} \in \mathbb{R}.$$

The first subalgebras contain ideal  $\langle e_5 \rangle$ , by factorization with the ideal we get the case considered above (Theorem III.2.1, and 2). That is why below we will examine only the subalgebras of the second kind which do not contain any ideals of the Lie algebra  $\mathfrak{G}$ , and are not ideals themselves (That is why the description of correspondent algebras is not reduced to the case  $\dim \mathfrak{G} = 4$ ).

Let us note that the automorphism  $A$  of Lie algebra  $\mathfrak{G}$ , which is the extension of an automorphism of the Lie triple system  $\mathfrak{G}$  (with the trilinear operation Type II [11]), is defined as follows:

$$ad(\xi) = \begin{pmatrix} bf^2 & a & d & 0 & 0 \\ 0 & b & l & 0 & 0 \\ 0 & 0 & f & 0 & 0 \\ 0 & 0 & 0 & bf & 0 \\ 0 & 0 & 0 & -af & bf^3 \end{pmatrix},$$

$b, f \neq 0, \quad a, d, l \in \mathbb{R}$   
so that

$$\begin{aligned} A(e_4 + \alpha e_1 + \beta e_2) &= e_4 + \alpha_n e_1 + \beta_n e_2, \\ A(e_5 + \bar{\alpha} e_1 + \bar{\beta} e_2) &= e_4 + \bar{\alpha}_n e_1 + \bar{\beta}_n e_2, \end{aligned}$$

where,

$$\begin{aligned} \alpha_n &= \alpha f + \frac{\beta}{f} \frac{a}{b} + \frac{\bar{\beta}}{f^2} \frac{a^2}{b^2}, \\ \beta_n &= \frac{\beta}{f} + \frac{\bar{\beta}}{f^3} \frac{a}{b}, \\ \bar{\alpha}_n &= \frac{\bar{\alpha}}{f} + \frac{\bar{\beta}}{f^3} \frac{a}{b}, \\ \bar{\beta}_n &= \frac{\bar{\beta}}{f^3}. \end{aligned}$$

- I. if  $\bar{\beta} = 0$  then  $\bar{\beta}_n = 0$  since  $\bar{\alpha} \neq 0$  otherwise we will get an ideal, by factorization through it, we will get the above case. Therefore we can make  $\bar{\alpha}_n = 1$  then we will have two possibilities:
  - $\beta_n = -1$  and  $\alpha_n = \alpha f + \frac{\beta}{\alpha} \frac{a}{b}$  is any number
  - $\beta_n = \frac{\beta}{\alpha}$  is any number not equal to  $-1$  and we make  $\alpha_n = 0$ .
- II. If  $\bar{\beta} \neq 0$ , then  $\bar{\beta}_n = 1$ 
  - If  $\frac{a}{b} = -\frac{\beta}{f}$ , then  $\beta_n = 0$ ,  $\bar{\alpha}_n = \frac{\bar{\alpha}}{f} + \frac{a}{b}$  is any number, and  $\alpha_n = \alpha f + \frac{a}{b} \frac{\beta}{f} + \frac{a}{b} \frac{\bar{\alpha}}{f} + (\frac{a}{b})^2 f$  is any number;
  - if  $\beta = 0$ , then  $\beta_n = \frac{a}{b}$  we can take  $a = 0$ ,  $\beta_n = 0$ , then  $\bar{\alpha}_n = 0$ ,  $\beta_n = \alpha f$  any number.

Here we note that the situation  $b$  above is the particular case of the situation  $a$  hence within the isomorphism there are

$$(\nu, -1, 1, 0), (0, \beta_n = \frac{\beta}{\bar{\alpha}} \neq -1, 0)$$

$$(\mu, 0, \theta, 1) \quad (4)$$

where

$$\nu = \alpha f + \frac{\beta}{\alpha} \frac{a}{b}, \mu = \alpha f + \frac{\beta}{\alpha} \frac{a}{b} + \frac{\bar{\alpha}}{f} \frac{a}{b} + (ab)^2 f, \theta = \frac{\bar{\alpha}}{f} + \frac{a}{b}.$$

The elements (4) form a section of orbits which is obtained under the action of the automorphism  $A$  in the 4-dimensional space  $(\alpha, \beta, \bar{\alpha}, \bar{\beta})$  respectively.

The following proposition holds

**Theorem III.2.3** Any Bol algebra of dimension 3, with the trilinear operation of Type II, and the canonical enveloping Lie algebra of dimension 5, is isomorphic to one of the Bol algebras of the form (4)

1.

$$\begin{aligned} e_2 \cdot e_3 &= -\nu e_1 + e_2, \quad -\nu \geq 0, \\ e_1 \cdot e_3 &= e_1 \quad (e_2, e_3, e_3) = e_1, \end{aligned}$$

2.

$$\begin{aligned} e_2 \cdot e_3 &= -\frac{\beta}{\bar{\alpha}} e_2, \quad -\frac{\beta}{\bar{\alpha}} \geq 0, \beta \neq -\bar{\alpha}, \\ e_1 \cdot e_3 &= e_1 \quad (e_2, e_3, e_3) = e_1, \end{aligned}$$

3.

$$\begin{aligned} e_2 \cdot e_3 &= -\mu e_1, \quad -\mu \geq 0, \\ e_1 \cdot e_3 &= -\theta e_1 - e_2, \quad \theta \geq 0 \quad (e_2, e_3, e_3) = e_1. \end{aligned}$$

**Remark.** The classification of Bol algebra up to the isotopy is not examined here.

Below, we give an example of 3-Web corresponding to the case  $(0, \beta, 1, 0)$  where  $\beta \neq -1$ .

Bol algebra  $\mathfrak{B}$  with the trilinear and bilinear operations defined as follows:

$$\begin{aligned} e_2 \cdot e_3 &= \beta e_2, \\ e_1 \cdot e_3 &= e_1 \quad (e_2, e_3, e_3) = e_1 \end{aligned}$$

has a 4-dimensional canonical enveloping Lie algebra of dimension 5,  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4, e_5 \rangle$ ,  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}$ ,  $\mathfrak{h} = \langle e_4 - \beta e_2, e_5 - e_1 \rangle$ .

The composition law  $(\Delta)$ , corresponding to the Lie group  $G$ , is defined as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \Delta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + \frac{x_4 y_3 - x_3 y_4}{2} - \frac{x_2 x_3 y_3 - x_3^2 y_2}{12} + \frac{x_3 y_2 y_3 - x_2 y_3^2}{12} \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 + \frac{x_2 y_3 - y_2 x_3}{2} \\ x_5 + y_5 + \frac{x_1 y_3 - x_3 y_1}{2} - \frac{x_3 x_4 y_3 - x_3^2 y_4}{12} + \frac{x_3 y_3 y_4 - x_4 y_3^2}{12} \end{bmatrix}.$$

The subgroup  $H$  is defined as the collection of elements

$$H = \exp \mathfrak{h} = \{\exp t(e_4 - \beta e_2), \exp q(e_5 - e_1)\}_{t,q \in \mathbb{R}} = \{-q, -t\beta, 0, t, q\}_{t,q \in \mathbb{R}}.$$

The collection of elements

$$B = \{t, u, v, 0, 0\}_{t,u,v \in \mathbb{R}}$$

form a local section of left coset space  $G \bmod H$ .

Any element  $(x_1, x_2, x_3, x_4, x_5)$  from  $G$ , can be uniquely represented as follows:

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} &= \begin{bmatrix} \frac{T}{6(1+x_3)(2+x_3)} \\ \frac{2\beta x_4}{2+\beta x_3} + x_2 \\ x_3 \\ 0 \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} \frac{x_4 \cdot x_3^2}{6(2+\beta x_3)} - x_5 \\ -\frac{2\beta x_4}{2+\beta x_3} \\ 0 \\ \frac{2x_4}{2+\beta x_3} \\ -\frac{x_4 \cdot x_3^2}{6(2+\beta x_3)} + x_5 \end{bmatrix} \\ &= \prod_B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \triangleq \begin{bmatrix} \frac{x_4 \cdot x_3^2}{6(2+\beta x_3)} - x_5 \\ -\frac{2\beta x_4}{2+\beta x_3} \\ 0 \\ \frac{2x_4}{2+\beta x_3} \\ -\frac{x_4 \cdot x_3^2}{6(2+\beta x_3)} + x_5 \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} T = 12x_1 + 6\beta x_1 x_3 + 12x_1 x_3 + 6\beta x_1 (x_3)^2 + 12x_5 + 6\beta x_3 x_5 - 6(x_3)^2 x_4 + \\ + 6x_3 x_4 + \beta x_4 (x_3)^2 + \beta (x_3)^2 x_4. \end{aligned} \quad (7)$$

The composition law  $(\star)$  of local analytic Bol loop  $B(\star)$  is defined as:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \prod_B \left( \begin{bmatrix} t + t' - \frac{uvv' - u'v^2}{12} - \frac{vu'v' - u(v')^2}{12} \\ u + u' \\ v + v' \\ \frac{uv' - u'v}{2} \\ \frac{tv' - vt'}{2} \end{bmatrix} \right) \\ &= \begin{bmatrix} X \\ u + u' + \frac{\beta(uv' - u'v)}{2 + \beta(v + v')} \\ v + v' \end{bmatrix}. \end{aligned}$$

Where

$$\begin{aligned}
X = t + t' - \frac{uvv' - vu'}{12} - \frac{vu'v' - u(v')^2}{12} + \frac{tv' - vt'}{2} - \frac{(uv' - vu')(v + v')}{2} + \\
+ \left( \frac{v + v'}{2} + \frac{\beta(v + v')^2}{12} \right) \frac{(uv' - vu')}{2 + \beta(v + v')}. \quad (8)
\end{aligned}$$

The corresponding local 3-Webs can be realized as a 6-order space.

### 4.3 BOL ALGEBRAS WITH TRILINEAR OPERATION OF TYPE III

As in the previous chapter we will base our investigation of 3-dimensional Bol algebras, on the examination of their canonical enveloping Lie algebras. In what follows, we consider Bol algebras of dimension 3, from their construction see [34,36]; it follows that the dimension of their canonical enveloping Lie algebras can not be more than 6. Below we limit ourselves to the classification of Bol algebras (and their corresponding 3-webs) with canonical enveloping Lie algebras of dimension  $\leq 4$ .

Let  $\mathfrak{B}$  be a 3-dimensional Bol algebra with a trilinear operation Type III see chapter II §3, and  $\mathfrak{G} = \mathfrak{B} + \mathfrak{h}$ -its canonical enveloping Lie algebra according to the table given in chapter II (case 4). We note that the situation  $\dim \mathfrak{G} = 3$  is possible that means we obtain a total grouped 3-Web the corresponding Lie group  $G$ , is isomorphic to the matrix of the view:

$$\begin{pmatrix} 1 & 0 & 0 \\ -(y-z)\frac{e^2x-1}{2x} & \frac{e^2x+1}{2} & -\frac{e^2x-1}{2} \\ (y-z)\frac{e^2x-1}{2x} & -\frac{e^2x+1}{2} & \frac{e^2x-1}{2} \end{pmatrix},$$

with  $x, y, z \in \mathbb{R}$ .

Let us examine the case  $\dim \mathfrak{G} = 4$ , the structural constants of Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$ ,  $\mathfrak{B} = \langle e_1, e_2, e_3 \rangle$  are defined as follows:

$$[e_1, e_2] = e_4, \quad [e_2, e_4] = \mp e_1$$

in addition  $\mathfrak{G} = \mathfrak{B} \dot{+} [\mathfrak{B}, \mathfrak{B}]$ ,  $[\mathfrak{B}, \mathfrak{B}] = \langle e_4 \rangle$ .

By introducing in consideration the 3-dimensional subspaces of subalgebras

$$\mathfrak{h}_{x,y,z} = \langle e_4 + xe_1 + ye_2 + ze_3 \rangle, \quad x, y, z, \in \mathbb{R}$$

we obtain a collection of Bol algebras of view:

\*\*\*\*\*

if  $y \neq 0$  then choosing  $\alpha, a$  and  $c$  one can make  $y' = 1$ ,  $x' = 0$ ,  $z' = 0$ .

If  $y = 0$ , then  $y' = 0$ , choose  $x' = \pm x \geq 0$ , and  $z' = 0, 1$ .

Thus within the isomorphism are obtained two families of Bol algebras and one exceptional Bol algebra.

**Theorem III .3.1** Any Bol algebra of dimension 3, with the trilinear operation of Type  $III^-$ , and the canonical enveloping Lie algebra of dimension 4, is isomorphic to one of Bol algebras below:

- $III^-.1$   $e_1 \cdot e_2 = -e_2$ ,  $(e_1, e_2, e_2) = e_1$ ,
- $III^-.1$   $e_1 \cdot e_2 = -xe_1$ ,  $(e_1, e_2, e_2) = e_1$   $x \geq 0$ ,
- $III^-.1$   $e_1 \cdot e_2 = -xe_1 - e_3$ ,  $(e_1, e_2, e_2) = e_1$   $x \geq 0$ .

In this the distinguished Bol algebras are not isomorphic among themselves.

Similarly one can establish the correctness of the following Theorem.

**Theorem III.3.2** Any Bol algebra of dimension 3, with the trilinear operation of Type  $III^+$ , and the canonical enveloping Lie algebra of dimension 4, is isomorphic to one of Bol algebras below:

- $III^+.1$   $e_1 \cdot e_2 = -e_2$ ,  $(e_1, e_2, e_2)- = e_1$ ,
- $III^+.1$   $e_1 \cdot e_2 = -xe_1$ ,  $(e_1, e_2, e_2)- = e_1$   $x \geq 0$ ,
- $III^+.1$   $e_1 \cdot e_2 = -xe_1 - e_3$ ,  $(e_1, e_2, e_2)- = e_1$   $x \geq 0$ .

In this the distinguished Bol algebras are not isomorphic among themselves.

We pass to the classification within isotopic of Bol algebras, enumerated in Theorem III.31.

We will note that for every  $\xi = te_1 + ue_2 + ve_3$  from  $\mathfrak{B}$   $t, u, v, \in \mathbb{R}$

$$ad(\xi) = \begin{pmatrix} 0 & 0 & 0 & -u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -u & t & 0 & 0 \end{pmatrix},$$

$$Ad(\xi) = \begin{pmatrix} \cosh u & \frac{t}{u}(1 - \cosh u) & 0 & -\sinh u \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh u & \frac{t}{u} \sinh u & 0 & \cosh u \end{pmatrix}.$$

Let us calculate the image of  $\Phi(\mathfrak{h})$  under the action of  $\Phi = Ad\xi$  on the one-dimensional subalgebra  $\mathfrak{h}$  with a directing vector

$$e_4 + xe_1 + ze_3, x \geq 0; 1$$

$$\Phi(e_4 + xe_1 + ze_3) = (\cosh u - x \sinh u)e_4 + ze_3 + (x \cosh u - \sinh u)e_1,$$

in addition one can define the mapping

$$x \longrightarrow \frac{x \cosh u - \sinh u}{-x \sinh u + \cosh u} = x',$$

$$z \longrightarrow \frac{z}{-x \sinh u + \cosh u} = z'.$$

By choosing  $u$  such that  $\tanh u = x$ , we obtain  $x' = 0$ . We notice that in addition  $\coth u \neq 0$ , that is, the mapping is defined correctly.

By applying the automorphism (9) to the so obtained Bol algebras, one can consider  $z' = 0; 1$ . Moreover, the two isolated case are not isotopic (in



the sense of the definition see chapter I). By virtue of the formulas (5) and (9).

We note that the application of the isotopic transformation  $\Phi$  to the exceptional Bol algebra of Theorem III.3.1 is not changed.

Summarizing the conducted examination one can formulate the Theorem  
**Theorem III .3.3** Any Bol algebra of dimension 3, with the trilinear operation of Type  $III^-$ , and the canonical enveloping Lie algebra  $\mathfrak{G}$  of dimension 4, is isotopic to one of the following Bol algebras:

- $e_1 \cdot e_2 = -e_2, (e_1, e_2, e_2) = e_1,$
- $e_1 \cdot e_2 = -ze_1, (e_1, e_2, e_2) = e_1 \quad z = 0; 1.$

Analogically one can state the correctness of the Theorem.

**Theorem III .3.4** Any Bol algebra of dimension 3, with the trilinear operation of Type  $III^+$ , and the canonical enveloping Lie algebra  $\mathfrak{G}$  of dimension 4, is isotopic to one of Bol algebras below:

- $e_1 \cdot e_2 = -e_2, (e_1, e_2, e_2)^- = e_1,$
- $e_1 \cdot e_2 = -ze_1, (e_1, e_2, e_2)^- = e_1 \quad x = 0, 1.$

Below we reduce to description of 3-Webs corresponding to the isolated Bol algebras of Type  $III^-$  and Type  $III^+$ .

The composition law ( $\Delta$ ), corresponding to the Lie group  $G$  of Lie algebra enveloping Bol algebra is defined as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Delta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + \cosh x_2 - y_4 \sinh x_2 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 - y_1 + \sinh x_2 + y_4 \cosh x_2 \end{bmatrix}.$$

In case  $III^-.1$  the subgroup  $H = \exp \mathfrak{h}$  can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{\exp \alpha(e_4 + e_2)\}_{\alpha \in \mathbb{R}} = \{0, \alpha, 0, \alpha\}_{\alpha \in \mathbb{R}}.$$

The collection of elements

$$B = \exp \mathfrak{B} = \left\{ \frac{t}{u} \sinh u, u, v, \frac{t}{u}(1 - \cosh u) \right\}_{t, u, v \in \mathbb{R}}$$

form a local section of left space coset  $G \bmod H$ .

$$\exp^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{x_1 x_2}{\sinh x_2} \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

$x_1, x_2, x_3, x_4 \in \mathbb{R}$ .

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , in the neighborhood  $e$ , can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} (x_1 \cosh p + x_4 \sinh p) \frac{p}{\sinh p} \\ p \\ x_3 \\ (x_1 \cosh p + x_4 \sinh p) \frac{1 - \cosh p}{\sinh p} \end{bmatrix} \triangle \begin{bmatrix} 0 \\ (x_4 - x_1 (\frac{1}{\sinh u} - \coth u)) \\ (x_4 - x_1 (\frac{1}{\sinh u} - \coth u)) \end{bmatrix},$$

where  $p$  is defined from the relation

$$p + x_4 - x_1 \frac{1 - \cosh p}{\sinh p} = x_2.$$

The composition law  $(\star)$ , corresponding to the local analytical Bol loop  $B(\star)$ , is defined as follows:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \triangle \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \right) \\ &= \exp^{-1} \left( \begin{bmatrix} [\cosh p + t' \cosh(u - p)] \frac{p}{\sinh p} \\ p \\ v + v' \\ [t \cosh p + t' \cosh(u - p)] \frac{(1 - \cosh p)}{\sinh p} \end{bmatrix} \right) \\ &= \begin{bmatrix} [t \cosh p + t' \cosh(u - p)] (\frac{p}{\sinh p})^2 \\ p \\ v + v' \end{bmatrix}, \end{aligned}$$

where  $p$  is defined from the relation:

$$p + \frac{-t' \sinh u \sinh p - (t + t' \cosh u)(1 - \cosh p)}{\sinh p} = u + u'.$$

The corresponding local analytic 3-Web, can be realized as a hyperbolic space.

In case  $III^-$ .2. the subgroup  $H = \exp \mathfrak{h}$  can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{\exp \alpha(e_4 + x e_1)\}_{\alpha \in \mathbb{R}} = \{x\alpha, 0, 0, \alpha\}_{\alpha \in \mathbb{R}}.$$

Bol loop can be realized from the elements

$$B = \exp \mathfrak{B} = \left\{ \frac{t}{u} \sinh u, u, v, \frac{t}{u} (1 - \cosh u) \right\}_{t, u, v \in \mathbb{R}}$$

which form a local section of left space coset  $G \bmod H$ .

$$\exp^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{x_1 x_2}{\sinh x_2} \\ x_2 \\ x_3 \end{bmatrix} \quad x_1, x_2, x_3, x_4 \in \mathbb{R}$$

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , in the neighborhood  $e$ , can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} \frac{[x_1(x \sinh x_2 - \cosh x_2) + x_4(x \cosh x_2) - \sinh x_2] \sinh x_2}{x(-1 + \cosh x_2) - \sinh x_2} \\ x_2 \\ x_3 \\ \frac{[x_1(x \sinh x_2 - \cosh x_2) + x_4(x \cosh x_2) - \sinh x_2](1 - \cosh x_2)}{x(-1 + \cosh x_2) - \sinh x_2} \end{bmatrix} \triangleq \begin{bmatrix} \frac{x(x_1(1 - \cosh x_2) - x_4 \sinh x_2)}{x(-1 + \cosh x_2) - \sinh x_2} \\ 0 \\ 0 \\ \frac{x(x_1(1 - \cosh x_2) + x_4 \sinh x_2)}{x(-1 + \cosh x_2) - \sinh x_2} \end{bmatrix}. \quad (9)$$

The composition law  $law(\star)$  corresponding to the local analytical Bol loop  $B(\star)$  is defined as follows:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \begin{bmatrix} t + t' \\ u + u' \\ v + v' \\ -t' \sinh u \end{bmatrix} \right) \\ &= \exp^{-1} \begin{bmatrix} \frac{[(t+t' \cosh u)(x \sinh(u+u') - \cosh(u+u')) - t' \sinh u(x \cosh(u+u') - \sinh(u+u'))] \sinh(u+u')}{x(\cosh(u+u') - 1) - \sinh(u+u')} \\ u + u' \\ v + v' \\ \frac{[(t+t' \cosh u)(x \sinh(u+u') - \cosh(u+u')) - t' \sinh u(x \cosh(u+u') - \sinh(u+u'))](1 - \cosh(u+u'))}{x(\cosh(u+u') - 1) - \sinh(u+u')} \end{bmatrix} \\ &= \begin{bmatrix} \frac{L_1(u+u')}{L_2} \\ u + u' \\ v + v' \end{bmatrix}, \end{aligned}$$

where

$$L_1 = [(t + t' \cosh u)(x \sinh(u + u') - \cosh(u + u')) - t' \sinh u(x \cosh(u + u') - \sinh(u + u'))], \quad (10)$$

$$L_2 = x(\cosh(u + u') - 1) - \sinh(u + u').$$

The corresponding local analytic 3-Web, can be realized as a hyperbolic space type.

In case  $III^-$ .3. the subgroup  $H = \exp \mathfrak{h}$  can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{ \exp \alpha(e_4 + xe_1) \}_{\alpha \in \mathbb{R}} = \{ x\alpha, 0, 0, \alpha \}_{\alpha \in \mathbb{R}}.$$

Bol loop can be realized from the elements

$$B = \exp \mathfrak{B} = \left\{ \frac{t}{u} \sinh u, u, v, \frac{t}{u}(1 - \cosh u) \right\}_{t, u, v \in \mathbb{R}}$$

which form a local section of left space coset  $G \bmod H$ .

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , in the neighborhood  $e$ , can be uniquely represented as follows:

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In case  $III^+$ .1 the subgroup  $H = \exp \mathfrak{h}$ , can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{ \exp \alpha(e_4 + e_2) \}_{\alpha \in \mathbb{R}} = \{ 0, \alpha, 0, \alpha \}_{\alpha \in \mathbb{R}}.$$

The collection of elements

$$B = \exp \mathfrak{B} \left\{ \frac{t}{u} \sinh u, u, v, \frac{t}{u}(1 - \cosh u) \right\}_{t, u, v \in \mathbb{R}}$$

form a local section of left space coset  $G \bmod H$ .

$$\exp^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{x_1 x_2}{\sinh x_2} \\ x_2 \\ x_3 \end{bmatrix} \quad x_1, x_2, x_3, x_4 \in \mathbb{R}.$$

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , in the neighborhood  $e$ , can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} (x_1 \cos p + x_4 \sin p) \frac{p}{\sin p} \\ p \\ x_3 \\ (x_1 \cos p + x_4 \sin p) \frac{1 - \cos p}{\sin p} \end{bmatrix} \triangle \begin{bmatrix} 0 \\ (x_4 - x_1(\frac{1}{\sin u} - \cot u)) \\ (x_4 - x_1(\frac{1}{\sin u} - \cot u)) \end{bmatrix},$$

where  $u$  is defined from the relation

$$u + x_4 - x_1(\cot u - \csc u) = x_2.$$

The composition law  $(\star)$ , corresponding to the local analytical Bol loop  $B(\star)$ , is defined as follows:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \begin{bmatrix} t + t' \\ u + u' \\ v + v' \\ -t' \sin u \end{bmatrix} \right) \\ &= \exp^{-1} \left( \begin{bmatrix} [t \cos p + t' \cos(u + p)] \frac{p}{\sin p} \\ p \\ v + v' \\ [t \cos p + t' \cos(u + p)] \frac{(1 - \cos p)}{\sin p} \end{bmatrix} \right) \\ &= \begin{bmatrix} [t \cos p + t' \cos(u + p)] \left( \frac{p}{\sin p} \right)^2 \\ p \\ v + v' \end{bmatrix}, \end{aligned}$$

where  $p$  is defined from the relation:

$$p + \frac{-t' \sin u \sin p - (t + t' \cos u)(1 - \cos p)}{\sin p} = u + u'.$$

In case  $III^+$ .2. The subgroup  $H = \exp \mathfrak{h}$ , can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{\exp \alpha(e_4 + x e_1)\}_{\alpha \in \mathbb{R}} = \{x\alpha, 0, 0, \alpha\}_{\alpha \in \mathbb{R}}.$$

Bol loop can be realized from the elements

$$B = \exp \mathfrak{B} = \left\{ \frac{t}{u} \sin u, u, v, \frac{t}{u}(1 - \cos u) \right\}_{t, u, v \in \mathbb{R}}$$

which form a local section of left space coset  $G \bmod H$ .

$$\exp^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{x_1 x_2}{\sin x_2} \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1, x_2, x_3, x_4 \in \mathbb{R}$ .

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$  in the neighborhood  $e$  can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} x_1 - \frac{(x \cos x_2 - \sin x_2)(x_1(1 - \cos x_2) + x_4 \sin x_2)}{x(1 - \cos x_2) - \sin x_2} \\ x_2 \\ x_3 \\ x_4 - \frac{(\cos x_2 - x \sin x_2)(x_1(1 - \cos x_2) + x_4 \sin x_2)}{x(1 - \cos x_2) - \sin x_2} \end{bmatrix} \triangleq \begin{bmatrix} \frac{x(x_1(1 - \cos x_2) + x_4 \sin x_2)}{x(1 - \cos x_2) - \sin x_2} \\ 0 \\ 0 \\ \frac{x(x_1(1 - \cos x_2) + x_4 \sin x_2)}{x(1 - \cos x_2) - \sin x_2} \end{bmatrix}.$$

The composition law  $(\star)$ , corresponding to the local analytical Bol loop  $B(\star)$ , is defined as follows:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \begin{bmatrix} t + t' \cos u \\ u + u' \\ v + v' \\ -t' \sin u \end{bmatrix} \right) \\ &= \exp^{-1} \begin{bmatrix} t + t' \cos u - \frac{N_1}{x(1-\cos(u+u'))-\sin(u+u')} \\ u + u' \\ v + v' \\ -t' \sin u - \frac{N_2}{x(1-\cos(u+u'))-\sin(u+u')} \end{bmatrix} \\ &= \begin{bmatrix} \left[ t + t' \cos u - \frac{N_1}{x(1-\cos(u+u'))-\sin(u+u')} \right] \frac{u+u'}{\sin(u+u')} \\ u + u' \\ v + v' \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} N_1 &= (x \cos(u+u') + \sin(u+u'))((t+t' \cos u)(\cos(u+u')-1) + t' \sin u \sin(u+u')), \\ N_2 &= (\cos(u+u') + x \sin(u+u'))((t+t' \cos u)(\cos(u+u')-1) + t' \sin u \sinh(u+u')). \end{aligned}$$

In case  $III^+$ .3. The subgroup  $H = \exp \mathfrak{h}$ , can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{\exp \alpha(e_4 + x e_1 + e_3)\}_{\alpha \in \mathfrak{R}} = \{x\alpha, 0, \alpha, \alpha\}_{\alpha \in \mathbb{R}}$$

Bol loop can be realized from the elements

$$B = \exp \mathfrak{B} = \left\{ \frac{t}{u} \sin u, u, v, \frac{t}{u}(1 - \cos u) \right\}_{t, u, v \in \mathbb{R}}$$

which form a local section of left space coset  $G \bmod H$ .

$$\exp^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{x_1 x_2}{\sin x_2} \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1, x_2, x_3, x_4 \in \mathbb{R}$ .

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , in the neighborhood  $e$ , can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} x_1 - \frac{(x \cos x_2 + \sin x_2)(x_1(-1 + \cos x_2) + x_4 \sin x_2)}{x(1 - \cos x_2) + \sin x_2} \\ x_2 \\ x_3 - \frac{x(x_4 \sin x_2 - x_1(\cos x_2 - 1))}{x(1 - \cos x_2) + \sin x_2} \\ x_4 - \frac{(\cos x_2 - x \sin x_2)(x_1 \cos x_2) + x_4 \sin x_2}{x(1 - \cos x_2) + \sin x_2} \end{bmatrix} \triangleq \begin{bmatrix} \frac{x(-x_1(-1 + \cos x_2) + x_4 \sin x_2)}{x(1 - \cos x_2) - \sin x_2} \\ 0 \\ \frac{(-x_1(-1 + \cos x_2) + x_4 \sin x_2)}{x(1 - \cos x_2) - \sin x_2} \\ \frac{(x_1(-1 + \cos x_2) + x_4 \sin x_2)}{x(1 - \cos x_2) + \sin x_2} \end{bmatrix}.$$

The composition law  $(\star)$ , corresponding to the local analytical Bol loop  $B(\star)$ , is defined as follows:

$$\begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} = \exp^{-1} \left( \prod_B \begin{bmatrix} t + t' \cos u \\ u + u' \\ v + v' \\ -t' \sin u \end{bmatrix} \right) \\ = \begin{bmatrix} t + t' \cos u - \frac{P_1}{x(1-\cos(u+u'))+\sin(u+u')} \frac{u+u'}{\sin(u+u')} \\ v + v' + \frac{t' \sin u \sin(u+u')+(t+t' \cos u)(\cos(u+u')-1)}{x(1-\cos(u+u'))+\sin(u+u')} \end{bmatrix},$$

where

$$P_1 = (x \cos(u+u')+\sin(u+u'))(t' \sinh u \sin(u+u')-(t+t' \cos u)(\cos(u+u')-1)).$$

#### 4.4 BOL ALGEBRAS WITH TRILINEAR OPERATION OF TYPE IV

As in the previous chapter we will base our investigation of 3-dimensional Bol algebras on the examination of their canonical enveloping Lie algebras. In what follows, we consider Bol algebras of dimension 3, from their construction see [30, 34]. Below we limit ourselves to the classification of Bol algebras (and their corresponding 3-Webs) with canonical enveloping Lie algebras of dimension  $\leq 4$ .

Let  $\mathfrak{B}$  be a 3-dimensional Bol algebra with a trilinear operation Type IV see chapter II §3, and  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}$  its canonical enveloping Lie algebra. According to the table given in chapter II(case 4). We note that the situation  $\dim \mathfrak{G} = 3$  is impossible, that means the case of grouped 3-Webs is excluded.

Let us examine the case  $\dim \mathfrak{G} = 4$ , the structural constants of Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$ ,  $\mathfrak{B} = \langle e_1, e_2, e_3 \rangle$  are defined as follows:

$$[e_1, e_2] = e_4, \quad [e_2, e_4] = \mp e_1$$

(1)

$$[e_1, e_3] = \pm e_4, \quad [e_3, e_4] = -e_1$$

in addition  $\mathfrak{G} = \mathfrak{B} \dot{+} [\mathfrak{B}, \mathfrak{B}]$ ,  $[\mathfrak{B}, \mathfrak{B}] = \langle e_4 \rangle$ .

By introducing in consideration the 3-dimensional subspaces of subalgebras

$$\mathfrak{h}_{x,y,z} = \langle e_4 + xe_1 + ye_2 + ze_3 \rangle, \quad x, y, z, \in \mathbb{R}$$

we obtain a collection of Bol algebras of view:

$$e_1 \cdot e_2 = xe_1 + ye_2 + ze_3,$$

$$e_1 \cdot e_3 = \mp xe_1 \mp ye_2 \mp ze_3,$$

(2)

$$(e_1, e_2, e_3) = \pm e_1, \quad (e_1, e_3, e_2) = -e_1$$

$$(e_1, e_2, e_3) = e_1, \quad (e_1, e_3, e_2) = \mp e_1.$$

Our main problem will be to give, an isomorphical and isotopical classification of Bol algebras of view (2).

For the full examination of this case, we will split it into cases Type  $IV^-$  and Type  $IV^+$ , corresponding to the upper and the lower signs of the formulas (1) and (2).

The group of automorphisms  $F$  of Lie triple system  $\mathfrak{B}$  relatively to a fixed base  $e_1, e_2, e_3$  from Type  $IV^-$  is defined as follows:



$$F = \left\{ P = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \right\}. \quad (3)$$

The extension of automorphism  $A$  from  $F$  to the automorphism of Lie algebra  $\mathfrak{G}$ , transforming the subspace  $\mathfrak{B}$  into itself can be realized as follows:

$$\begin{aligned} Ae_4 &= A[e_1, e_2] = [e_1, Ae_2] = \mp e_4, \\ \text{or} \quad (4) \\ \mp Ae_4 &= A[e_1, e_3] = [Ae_1, Ae_3] = -\alpha e_4. \end{aligned}$$

In addition

$$A(e_4 + xe_1 + ye_2 + ze_3) = \pm \alpha e_4 + xe_1 \pm ye_2 \pm ze_3 = \pm \alpha (e_4 \pm xe_1 + \frac{y}{\alpha} e_2 + \frac{z}{\alpha} e_3),$$

that is

$$A(\mathfrak{h}_{x,y,z}) = \mathfrak{h}_{x',y',z'},$$

where

$$\begin{aligned} x' &= \pm x, \\ y' &= \frac{y}{\alpha}, \\ z' &= \frac{z}{\alpha}. \end{aligned} \quad (5)$$

If  $z \neq 0$  then at the expense of choice of  $\alpha$  one can make  $y' = p$ ,  $x' = \pm x \geq 0$ .

If  $z = 0$ , then at the expense of choice of  $\alpha$  one can make  $y' = 1$  and  $x' = \pm x \geq 0$  and  $y' = 1$ ,  $z' = 0$ .

Thus within the isomorphism we obtain two families of Bol algebras.

**Theorem  $IV^-$ .3.1** Any Bol algebra of dimension 3, with the trilinear operation of Type  $IV^-$  and the canonical enveloping Lie algebra of dimension 4, is isomorphic to one of Bol algebras below:

- $IV^-$ .1.  $e_1 \cdot e_2 = xe_1 + pe_2 + e_3$ ,  $(e_1, e_2, e_2) = e_1$ ,  $(e_1, e_3, e_2) = -e_1x \geq 0$   
 $e_1 \cdot e_2 = -xe_1 - pe_2 - e_3$ ,  $(e_1, e_2, e_3) = e_1$ ,  $(e_1, e_3, e_3) = -e_1x \geq 0$  for any  $p$ ,
- $IV^-$ .2.  $e_1 \cdot e_2 = xe_1 + pe_2$ ,  $(e_1, e_2, e_2) = e_1$ ,  $(e_1, e_3, e_2) = -e_1x \geq 0$   
 $e_1 \cdot e_2 = -xe_1 - pe_2$ ,  $(e_1, e_2, e_3) = e_1$ ,  $(e_1, e_3, e_3) = -e_1x \geq 0$  for any  $p$ .

the distinguished Bol algebras are not isomorphic among themselves.

Similarly one can establish the correctness of the following Theorem.

**Theorem  $IV^+$ .3.1** Any Bol algebra of dimension 3, with the trilinear operation of Type  $IV^+$  and the canonical enveloping Lie algebra of dimension 4, is isomorphic to one of Bol algebras below:

- $IV^+.1.e_1 \cdot e_2 = xe_1 + pe_2 + e_3$ ,  $(e_1, e_2, e_2) = -e_1$ ,  $(e_1, e_3, e_2) = -e_1x \geq 0$   
 $e_1 \cdot e_2 = -xe_1 - pe_2 - e_3$ ,  $(e_1, e_2, e_3) = e_1$ ,  $(e_1, e_3, e_3) = e_1x \geq 0$  for any  $p$ ,
- $IV^+.2.e_1 \cdot e_2 = xe_1 + pe_2$ ,  $(e_1, e_2, e_2) = -e_1$ ,  $(e_1, e_3, e_2) = -e_1x \geq 0$   
 $e_1 \cdot e_2 = -xe_1 - pe_2$ ,  $(e_1, e_2, e_3) = e_1$ ,  $(e_1, e_3, e_3) = e_1x \geq 0$ .

Also this distinguished Bol algebras are not isomorphic among themselves.

Let us pass to the isotopic classification of Bol algebras given in Theorems  $IV^-.3.1$ .

We note that for every  $\xi = ae_1 + be_2 + ce_3$  from  $\mathfrak{B}$   $a, b, c, \in \mathbb{R}$

$$ad(\xi) = \begin{pmatrix} 0 & 0 & 0 & -b-c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -b+c & a & -a & 0 \end{pmatrix},$$

$$Ad(\xi) = \begin{pmatrix} \cosh \sqrt{df} & \frac{a}{f}(-1 + \cosh \sqrt{df}) & \frac{a}{f}(1 - \cosh \sqrt{df}) & \sqrt{\frac{d}{f}} \sinh \sqrt{df} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{\frac{d}{f}} \sinh \sqrt{df} & \sqrt{\frac{a^2}{df}} \sinh \sqrt{df} & \sqrt{\frac{a^2}{df}} \sinh \sqrt{df} & \cosh \sqrt{df} \end{pmatrix}$$

where  $d = -b - c$ ,  $f = -b + c$ .

Let us find the image of  $\Phi(\mathfrak{h})$  under the action of  $\Phi = Ad\xi$  on the one-dimensional subalgebra  $\mathfrak{h}$  with a direction vector

$$e_4 + xe_1 + pe_2 + ze_3, \text{ or } e_4 + xe_1 + e_2.$$

Moreover by the local character of the consideration we will limit ourselves to the values  $a, b$  and  $c$  such that:

$$\cosh \sqrt{df} - \frac{a}{\sqrt{fd}} + x\sqrt{\frac{f}{d}} \sinh \sqrt{fd} + \frac{pa}{\sqrt{df}} \sinh \sqrt{df} \neq 0,$$

$$\begin{aligned} \Phi(e_4 + xe_1 + pe_2 + ze_3) &= (\cosh \sqrt{df} - \frac{a}{\sqrt{fd}} + x\sqrt{\frac{f}{d}} \sinh \sqrt{fd} + \frac{pa}{\sqrt{df}} \sinh \sqrt{df})e_4 + \\ &+ e_3 + pe_2 + (\sqrt{\frac{d}{f}} \sinh \sqrt{df} + x \cosh \sqrt{df} + p\frac{a}{f}(1 + \cosh \sqrt{fd}) + \frac{a}{f}(1 - \cosh \sqrt{df}))e_1 \end{aligned} \quad (11)$$

$$\begin{aligned}\Phi(e_4 + xe_1 + e_2) &= (\cosh \sqrt{df} + x\sqrt{\frac{f}{d}} \sinh \sqrt{fd} + \frac{a}{\sqrt{df}} \sinh \sqrt{df})e_4 + \\ &+ (\sqrt{\frac{d}{f}} \sinh \sqrt{df} + x \cosh \sqrt{df} + \frac{a}{f}(1 + \cosh \sqrt{fd}))e_1 + e_2\end{aligned}\quad (12)$$

Here we note that  $e_2$  and  $e_3$  are not changing only  $e_4$  and  $e_1$  are changing.

We denote

$$\begin{aligned}\tilde{\alpha} &= \cosh \sqrt{df} - \frac{a}{\sqrt{fd}} + x\sqrt{\frac{f}{d}} \sinh \sqrt{fd} + \frac{pa}{\sqrt{df}} \sinh \sqrt{df} \neq 0, \\ \tilde{\beta} &= \sqrt{\frac{d}{f}} \sinh \sqrt{df} + x \cosh \sqrt{df} + p\frac{a}{f}(1 + \cosh \sqrt{fd}) + \frac{a}{f}(1 - \cosh \sqrt{df}), \\ \Phi(\mathfrak{h}) &= \tilde{\alpha}e_4 + \tilde{\beta}e_1 + pe_2 + e_3 = \\ &= \tilde{\alpha} \left( e_4 + \frac{\tilde{\beta}}{\tilde{\alpha}} \right) e_1 + \frac{p}{\tilde{\alpha}}e_2 + \frac{1}{\tilde{\alpha}}.\end{aligned}$$

After the application of the automorphism  $A$  we obtain an expression of type:

$$\pm \frac{\tilde{\beta}}{\tilde{\alpha}}e_1 + pe_2 + e_3$$

The checking shows that  $\frac{\tilde{\beta}}{\tilde{\alpha}}$  are any values. By a nice choice of values  $a, b, c$ . One can note that, the application of the isotopic transformation  $\Phi$  to the Bol algebras of Theorem  $IV^-$  is not changing them. In what follows denote

$$\delta = \frac{\tilde{\beta}}{\tilde{\alpha}}.$$

Summarizing the conducted examination one can formulate the Theorem:

**Theorem  $IV^-$  .3.1** Any Bol algebra of dimension 3, with the trilinear operation of Type  $IV^-$  and the canonical enveloping Lie algebra  $\mathfrak{G}$  of dimension 4, is isotopic to one of the following Bol algebras:

- $e_1 \cdot e_2 = \pm \delta e_1 + pe_2 + e_3$ ,  $(e_1, e_2, e_2) = e_1$ ,  $(e_1, e_3, e_2) = -e_1$ ,  $\delta \geq 0$   
 $e_1 \cdot e_3 = \mp \delta e_1 - pe_2 - e_3$ ,  $(e_1, e_2, e_3) = e_1$ ,  $(e'_1 e_3, e_2) = -e_1$ ,  $\delta \geq 0$   
 where  $p$  is any number,
- $e_1 \cdot e_2 = \pm \delta e_1 + e_2$ ,  $(e_1, e_2, e_2) = e_1$ ,  $(e'_1 e_3, e_2) = -e_1$ ,  $\delta \geq 0$ ,  $e_1 \cdot e_3 = \mp \delta e_1 - e_2$ ,  $(e_1, e_2, e_3) = e_1$ ,  $(e'_1 e_3, e_2) = -e_1$ ,  $\delta \geq 0$ .

Analogically one can state the correctness of the Theorem:

**Theorem  $IV^+$  .3.1** Any Bol algebra of dimension 3, with the trilinear operation of Type  $IV^+$  and the canonical enveloping Lie algebra  $\mathfrak{G}$  of dimension 4, is isotopic to one of the following Bol algebras:

- $e_1 \cdot e_2 = \pm \delta e_1 + p e_2 + e_3$ ,  $(e_1, e_2, e_2) = -e_1$ ,  $(e'_1 e_3, e_2) = -e_1, \delta \geq 0$   
 $e_1 \cdot e_3 = \mp \delta e_1 - p e_2 - e_3$ ,  $(e_1, e_2, e_3) = e_1$ ,  $(e'_1 e_3, e_2) = e_1, \delta \geq 0$  where  
 $p$  is any number,
- $e_1 \cdot e_2 = \pm \delta e_1 + e_2$ ,  $(e_1, e_2, e_2) = -e_1$ ,  $(e'_1 e_3, e_2) = -e_1, \delta \geq 0$   $e_1 \cdot e_3 =$   
 $\mp \delta e_1 + e_2$ ,  $(e_1, e_2, e_3) = e_1$ ,  $(e'_1 e_3, e_2) = e_1$ .

Below we reduce to description of 3-Webs corresponding to the isolated Bol algebras of Type  $IV^-$  and Type  $IV^+$ . The composition law  $(\Delta)$ , corresponding to the Lie group  $G$  of Lie algebra enveloping Bol algebra is defined as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Delta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + \cosh(x_2 + x_3) - y_4 \sinh(x_2 + x_3) \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 - y_1 \sinh(x_2 + x_3) + y_4 \cosh(x_2 + x_3) \end{bmatrix}.$$

In case  $IV^-$ .1 the subgroup  $H = \exp \mathfrak{h}$  can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{\exp \alpha(e_4 + x e_1 + p e_2 + e_3)\}_{\alpha \in \mathbb{R}} = \{x\alpha, p\alpha, \alpha, \alpha\}_{\alpha \in \mathbb{R}}.$$

The collection of elements

$$B = \exp \mathfrak{B} = \left\{ \frac{t}{u+v} \sinh(u+v), u, v, \frac{t}{u+v} (1 - \cosh(u+v)) \right\}_{t, u, v \in \mathbb{R}}$$

form a local section of left space coset  $G \bmod H$ .

$$\exp^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{x_1(x_2+x_3)}{\sinh(x_2+x_3)} \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$x_1, x_2, x_3, x_4 \in \mathbb{R}$ .

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , in the neighbourhood  $e$ , can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} \frac{[x_1(\cosh(u+v) - x \sinh(u+v)) - x_4(x \cosh(u+v) - \sinh(u+v))] \sinh(u+v)}{x(1 - \cosh(u+v)) + \sinh(u+v)} \\ u \\ v \\ \frac{[x_1(\cosh(u+v) - x \sinh(u+v)) - x_4(x \cosh(u+v) - \sinh(u+v))] (\cosh(u+v) - 1)}{x(1 - \cosh(u+v)) + \sinh(u+v)} \end{bmatrix} \Delta \begin{bmatrix} x \frac{x_4 \sinh(u+v) + x_1(\cosh(u+v) - 1)}{x(1 - \cosh(u+v)) + \sinh(u+v)} \\ p \frac{x_4 \sinh(u+v) + x_1(\cosh(u+v) - 1)}{x(1 - \cosh(u+v)) + \sinh(u+v)} \\ \frac{x_4 \sinh(u+v) + x_1(\cosh(u+v) - 1)}{x(1 - \cosh(u+v)) + \sinh(u+v)} \\ \frac{x_4 \sinh(u+v) + x_1(\cosh(u+v) - 1)}{x(1 - \cosh(u+v)) + \sinh(u+v)} \end{bmatrix}, \quad (13)$$

where  $u + v$  are any numbers defined from the relation

$$u + v + (p + 1) \frac{x_1(\cosh(u + v) - 1) + x_4 \sinh(u + v)}{x(1 - \cosh(u + v)) + \sinh(u + v)} = x_2 + x_3.$$

The composition law  $(\star)$ , corresponding to the local analytical Bol loop  $B(\star)$ , is defined as follows:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \triangle \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \right) \\ \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t + t' \cosh(u + v) \\ u + u' \\ v + v' \\ -t \sinh(u + v) \end{bmatrix} \right) \right) \\ &= \exp^{-1} \left( \begin{bmatrix} T \sinh(m + n) \\ m \\ n \\ T(\cosh(m + n) - 1) \end{bmatrix} \right) \\ &= \begin{bmatrix} T(m + n) \\ m \\ n \end{bmatrix}, \end{aligned}$$

where  $T$  is defined from the relation:

$$T = \frac{D}{x(1 - \cosh(m + n) + \sinh(m + n))},$$

with

$$\begin{aligned} D &= (t + t' \cosh(u + v))(\cosh(m + n) - x \sinh(m + n)) + \\ &\quad + t \sinh(u + v)(x \cosh(m + n) - \sinh(m + n)), \end{aligned} \quad (14)$$

and  $m + n$  from the relation

$$m + n + (p + 1) \frac{(t + t' \cosh(u + v))(\cosh(m + n) - 1) - t \sinh(u + v) \sinh(m + n)}{x(1 - \cosh(m + n)) + \sinh(m + n)} = \Omega$$

and  $\Omega = u + u' + v + v'$ .

In case  $IV^- .2$  the subgroup  $H = \exp \mathfrak{h}$ , can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{ \exp \alpha(e_4 + x e_1 + e_2) \}_{\alpha \in \mathbb{R}} = \{ x \alpha, \alpha, 0, \alpha \}_{\alpha \in \mathbb{R}}.$$

The collection of elements

$$B = \exp \mathfrak{B} = \left\{ \frac{t}{u+v} \sinh(u+v), u, v, \frac{t}{u+v} (1 - \cosh(u+v)) \right\}_{t,u,v \in \mathbb{R}}$$

form a local section of left space coset  $G \bmod H$ .

$$\exp^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{x_1(x_2+x_3)}{\sinh(x_2+x_3)} \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1, x_2, x_3, x_4 \in \mathbb{R}$ .

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , in the neighbourhood  $e$ , can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} \frac{[x_1(\cosh(u+x_3) - x \sinh(u+x_3)) - x_4(x \cosh(u+x_3) - \sinh(u+x_3))] \sinh(u+x_3)}{x(1 - \cosh(u+x_3)) + \sinh(u+x_3)} \\ u \\ x_3 \\ \frac{[x_1(\cosh(u+x_3) - x \sinh(u+x_3)) - x_4(x \cosh(u+x_3) - \sinh(u+x_3))](\cosh(u+x_3) - 1)}{x(1 - \cosh(u+x_3)) + \sinh(u+x_3)} \end{bmatrix} \triangle$$

$$\begin{bmatrix} x \frac{x_4 \sinh(u+x_3) + x_1(\cosh(u+x_3) - 1)}{x(1 - \cosh(u+x_3)) + \sinh(u+x_3)} \\ \frac{x_4 \sinh(u+x_3) + x_1(\cosh(u+x_3) - 1)}{x(1 - \cosh(u+x_3)) + \sinh(u+x_3)} \\ 0 \\ \frac{x_4 \sinh(u+x_3) + x_1(\cosh(u+x_3) - 1)}{x(1 - \cosh(u+x_3)) + \sinh(u+x_3)} \end{bmatrix}, \quad (15)$$

where  $u$  are any numbers defined from the relation

$$u + x_3 + (p+1) \frac{x_1(\cosh(u+x_3) - 1) + x_4 \sinh(u+x_3)}{x(1 - \cosh(u+x_3)) + \sinh(u+x_3)} = x_2 + x_3.$$

The composition law  $(\star)$ , corresponding to the local analytical Bol loop  $B(\star)$ , is defined as follows:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \triangle \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \right) \\ \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t + t' \cosh(u+v) \\ u + u' \\ v + v' \\ -t \sinh(u+v) \end{bmatrix} \right) \right) \\ &= \exp^{-1} \left( \begin{bmatrix} [H \sinh(m + V + V')] \\ m \\ V + V' \\ [H(\cosh(m + v + v') - 1)] \end{bmatrix} \right) \end{aligned}$$

$$= \begin{bmatrix} H(m+v+v') \\ m \\ v+v' \end{bmatrix},$$

where  $H$  is defined from the relation:

$$H = \frac{A}{x(1 - \cosh(m+v+v') + \sinh(m+v+v'))},$$

$$A = (t + t' \cosh(u+v))(\cosh(m+v+v') - x \sinh(m+v+v')) + t \sinh(u+v+v')(x \cosh(m+v+v') - \sinh(m+v+v')), \quad (16)$$

and by denoting  $X = m+v+v'$ ,  $m$  will be defined from the relation

$$X + \frac{\Gamma}{x(1 - \cosh(m+v+v') + \sinh(m+v+v'))} = u + u',$$

$\Gamma = (t+t' \cosh(u+v+v'))(\cosh(m+v+v') - 1) - t \sinh(u+v+v') \sinh(m+v+v')$ .

We pass to the description of Bol 3-Webs corresponding to the Type  $IV^+$ .

The composition law ( $\triangle$ ), corresponding to Lie group  $G$ , with enveloping Lie algebra of Bol algebra of Type  $IV^+$  is defined as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \triangle \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + \cos(x_2 - x_3) - y_4 \sin(x_2 - x_3) \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 - y_1 \sin(x_2 - x_3) + y_4 \cos(x_2 - x_3) \end{bmatrix}.$$

In case  $IV^+.1$  the subgroup  $H = \exp \mathfrak{h}$ , can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{ \exp \alpha (e_4 + x e_1 + p e_2 + e_3) \}_{\alpha \in \mathbb{R}} = \{ x \alpha, p \alpha, \alpha, \alpha \}_{\alpha \in \mathbb{R}}.$$

The collection of elements

$$B = \exp \mathfrak{B} = \left\{ \frac{t}{u+v} \sin(u+v), u, v, \frac{t}{u+v} (1 - \cos(u+v)) \right\}_{t, u, v \in \mathbb{R}}$$

form a local section of left space coset  $G \bmod H$ .

$$\exp^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{x_1(x_2-x_3)}{\sin(x_2-x_3)} \\ x_2 \\ x_3 \end{bmatrix}.$$

$x_1, x_2, x_3, x_4 \in \mathbb{R}$

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , in the neighbourhood  $e$ , can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} \frac{[x_1(\cos(u-v)-x\sin(u-v))-x_4(x\cos(u-v)-\sin(u-v))]\sin(u-v)}{x(1-\cos(u-v))+\sin(u-v)} \\ u \\ v \\ \frac{[x_1(\cos(u-v)-x\sin(u-v))-x_4(x\cos(u-v)-\sin(u-v))](\cos(u-v)-1)}{x(1-\cos(u-v))+\sin(u-v)} \end{bmatrix} \triangleq \begin{bmatrix} x \frac{x_4 \sin(u-v)+x_1(\cos(u-v)-1)}{x(1-\cos(u-v))+\sin(u-v)} \\ p \frac{x_4 \sin(u-v)+x_1(\cos(u-v)-1)}{x(1-\cos(u-v))+\sin(u-v)} \\ \frac{x_4 \sin(u-v)+x_1(\cos(u-v)-1)}{x(1-\cos(u-v))+\sin(u-v)} \\ \frac{x_4 \sin(u-v)+x_1(\cos(u-v)-1)}{x(1-\cos(u-v))+\sin(u-v)} \end{bmatrix}, \quad (17)$$

where  $u - v$  are any numbers defined from the relation

$$u - v + (p + 1) \frac{x_1(\cos(u - v) - 1) + x_4 \sin(u - v)}{x(1 - \cos(u - v)) + \sin(u - v)} = x_2 + x_3.$$

The composition law  $(\star)$ , corresponding to the local analytical Bol loop  $B(\star)$ , is defined as follows:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \right) \\ \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t + t' \cos(u - v) \\ u + u' \\ v + v' \\ -t \sin(u - v) \end{bmatrix} \right) \right) \\ &= \exp^{-1} \left( \begin{bmatrix} [K \sin(m - n)] \\ m \\ n \\ [K(\cos(m - n) - 1)] \end{bmatrix} \right) \\ &= \begin{bmatrix} K(m - n) \\ m \\ n \end{bmatrix}, \end{aligned}$$

where  $K$  is defined from the relation:

$$K = \frac{\Delta}{x(1 - \cos(m - n) + \sin(m - n))},$$



$$\Delta = [(t + t' \cos(u - v))(\cos(m - n) - x \sin(m - n)) + t \sin(u - v)(x \cos(m - n) - \sin(m - n))], \quad (18)$$

and  $m - n$  from the relation

$$m - n + (p + 1) \frac{Y}{x(1 - \cos(m - n)) + \sin(m - n)} = u + u' + v + v',$$

$$Y = (t + t' \cos(u - v))(\cos(m - n) - 1) - t \sin(u - v) \sin(m - n).$$

In case  $IV^+$ . 2. The subgroup  $H = \exp \mathfrak{h}$ , can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{ \exp \alpha(e_4 + x e_1 + e_2) \}_{\alpha \in \mathbb{R}} = \{ x \alpha, \alpha, 0, \alpha \}_{\alpha \in \mathbb{R}}.$$

The collection of elements

$$B = \exp \mathfrak{B} = \left\{ \frac{t}{u - v} \sin(u - v), u, v, \frac{t}{u - v} (1 - \cos(u - v)) \right\}_{t, u, v \in \mathbb{R}}$$

form a local section of left space coset  $G \bmod H$ .

$$\exp^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{x_1(x_2 - x_3)}{\sinh(x_2 - x_3)} \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1, x_2, x_3, x_4 \in \mathbb{R}$ .

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , in the neighbourhood  $e$ , can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} \frac{[x_1(\cos(u - x_3) - x \sin(u - x_3)) - x_4(x \cos(u - x_3) - \sin(u - x_3))] \sin(u - x_3)}{x(1 - \cos(u - x_3)) + \sin(u - x_3)} \\ u \\ x_3 \\ \frac{[x_1(\cos(u - x_3) - x \sin(u - x_3)) - x_4(x \cos(u - x_3) - \sin(u - x_3))] (\cos(u - x_3) - 1)}{x(1 - \cos(u - x_3)) + \sin(u - x_3)} \end{bmatrix} \Delta$$

$$\begin{bmatrix} x \frac{x_4 \sin(u - x_3) + x_1(\cos(u - x_3) - 1)}{x(1 - \cos(u - x_3)) + \sin(u - x_3)} \\ \frac{x_4 \sin(u - x_3) + x_1(\cos(u - x_3) - 1)}{x(1 - \cos(u - x_3)) + \sin(u - x_3)} \\ 0 \\ \frac{x_4 \sin(u - x_3) + x_1(\cos(u - x_3) - 1)}{x(1 - \cos(u - x_3)) + \sin(u - x_3)} \end{bmatrix}, \quad (19)$$

where  $u$  are any numbers defined from the relation

$$u + x_3 + (p + 1) \frac{x_1(\cos(u - x_3) - 1) + x_4 \sin(u - x_3)}{x(1 - \cos(u - x_3)) + \sin(u - x_3)} = x_2 + x_3.$$

The composition law  $(\star)$  corresponding to the local analytical Bol loop  $B(\star)$  is defined as follows:

$$\begin{aligned}
\begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \triangle \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \right) \\
\begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t + t' \cos(u - v) \\ u + u' \\ v + v' \\ -t \sinh(u - v) \end{bmatrix} \right) \right) \\
&= \exp^{-1} \left( \begin{bmatrix} [S \sin(l - v - v')] \\ l \\ v + v' \\ S(\cos(l - v - v') - 1) \end{bmatrix} \right) \\
&= \begin{bmatrix} S(l - v - v') \\ l \\ v + v' \end{bmatrix},
\end{aligned}$$

where  $S$  is defined from the relation:

$$S = \frac{W}{x(1 - \cos(l - v - v') + \sin(l - v - v'))}$$

with

$$\begin{aligned}
W &= (t + t' \cos(u - v))(\cos(l - v - v') - x \sin(l - v - v')) + \\
&\quad + t \sin(u - v - v')(x \cos(l - v - v') - \sin(l - v - v')), \quad (20)
\end{aligned}$$

and  $l$  from the relation

$$\begin{aligned}
l + v + v' + \frac{Z}{x(1 - \cos(l - v - v') + \sin(l - v - v'))} &= u + u', \\
Z &= (t + t' \cos(u - v - v'))(\cos(l - v - v') - 1) - t \sin(u - v - v') \sin(l - v - v').
\end{aligned}$$

#### 4.5 BOL ALGEBRA WITH TRILINEAR OPERATION OF TYPE V

In what follows we consider Bol algebras of dimension 3, from their construction see [30,34] in this paragraph we base our investigation of 3-dimensional Bol algebras, on the examination of their canonical enveloping Lie algebras. As we already state it follows that the dimension of their canonical enveloping Lie algebras can not be more than 6. Below we limit ourselves to the classification of Bol algebras ( and their corresponding 3-Webs), with canonical enveloping Lie algebras of dimension  $\leq 4$ .

Let us examine the case  $\dim \mathfrak{G} = 4$ , the structural constants of Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$ ,  $\mathfrak{B} = \langle e_1, e_2, e_3 \rangle$  are defined as follows:

$$[e_2, e_3] = e_4, \quad [e_2, e_4] = -e_1 \quad (1)$$

$$[e_3, e_4] = \mp e_4.$$

in addition  $\mathfrak{G} = \mathfrak{B} \dot{+} [\mathfrak{B}, \mathfrak{B}]$ ,  $[\mathfrak{B}, \mathfrak{B}] = \langle e_4 \rangle$ .

By introducing in consideration the 3-dimensional subspaces of subalgebras

$$\mathfrak{h}_{x,y,z} = \langle e_4 + xe_1 + ye_2 + ze_3 \rangle, \quad x, y, z, \in \mathbb{R}$$

we obtain a collection of Bol algebras of view:

$$e_2 \cdot e_3 = -xe_1 - ye_2 - ze_3, \quad (2)$$

$$(e_2, e_3, e_2) = e_1,$$

$$(e_2, e_3, e_3) = \pm e_1.$$

Our main problem will be to give an isomorphical and isotopical classification of Bol algebras of view (2).

For the full examination of this case, we will split it in cases Type  $V^-$  and Type  $V^+$ , corresponding to the upper and the lower signs of the formulas (1) and (2).

The group of automorphisms  $F$  of Lie triple system  $\mathfrak{B}$  relatively to a fixed base  $e_1, e_2, e_3$  from Type  $IV^-$  is defined as follows:

$$F = \left\{ A = \begin{pmatrix} \pm bf^2 & fb & d \\ 0 & b & f \\ 0 & 0 & \pm 1 \end{pmatrix}, b \neq 0 \right\} \quad (3)$$

The extension of automorphism  $A$  from  $F$  to the automorphism of Lie algebra  $\mathfrak{G}$ , transforming the subspace  $\mathfrak{B}$  into itself can be realized as follows:

$$Ae_4 = A[e_2, e_3] = [Ae_2, Ae_3] = \pm be_4. \quad (4)$$

In addition

$$A(e_4 + xe_1 + ye_2 + ze_3) = \pm b \left( e_4 \pm \frac{yfb + zd \pm xbf^2}{b} e_1 \pm \frac{yb + zf}{b} e_2 + \frac{z}{b} e_3 \right),$$

that is

$$A(\mathfrak{h}_{x,y,z}) = \mathfrak{h}_{x',y',z'},$$

where

$$\begin{aligned} x' &= \frac{yfb + zd \pm xbf^2}{b}, \\ y' &= \frac{yb + zf}{b}, \\ z' &= \frac{z}{b}. \end{aligned} \quad (5)$$

- If  $z \neq 0$ , then by choosing  $b, f$  and  $d$  one can make  $x' = 0, y' = 0, z' = 0$ ;
- if  $z = 0$ , hence  $z' = 0$  one can choose  $y' = \pm y \geq 0$ , and make  $x' = 0$ .

In this way, we obtain one family and one exceptional Bol algebra.

**Theorem  $V^-$ .5.1** Any Bol algebra of dimension 3, with the trilinear operation of Type  $V^-$  and the canonical enveloping Lie algebra of dimension 4, is isomorphic to one of Bol algebras below:

- $V^-.1.e_2 \cdot e_3 = -e_3, (e_2, e_3, e_2) = e_1, (e_2, e_3, e_3) = e_2,$
- $V^-.2.e_2 \cdot e_3 = -ye_2, (e_2, e_3, e_2) = e_1, (e_2, e_3, e_3) = e_2, y \geq 0.$

The distinguished Bol algebras are not isomorphic among themselves. Similarly one can establish the correctness of the following Theorem.

**Theorem  $V^+$ .3.1** Any Bol algebra of dimension 3, with the trilinear operation of Type  $V^+$  and the canonical enveloping Lie algebra of dimension 4, is isomorphic to one of Bol algebras below:

- $V^+.1.e_2 \cdot e_3 = -e_3, (e_2, e_3, e_2) = e_1, (e_2, e_3, e_3) = e_2,$
- $V^+.2.e_2 \cdot e_3 = -ye_2, (e_2, e_3, e_2) = e_1, (e_2, e_3, e_3) = -e_2, y \geq 0.$

Also this distinguished Bol algebras are not isomorphic among themselves.

Let us pass to the isotopic classification of Bol algebras given in Theorems  $V^-.3.1$ .

We note that for every  $\xi = ue_1 + ve_2 + pe_3$  from  $\mathfrak{B}$   $u, v, p, \in \mathbb{R}$

$$ad(\xi) = \begin{pmatrix} 0 & 0 & 0 & -v \\ 0 & 0 & 0 & -p \\ 0 & 0 & 0 & 0 \\ 0 & -p & v & 0 \end{pmatrix},$$

$$Ad(\xi) = \begin{pmatrix} 1 & \frac{v(\cosh p - 1)}{p} & -\frac{(\cosh p - 1)v^2}{p^2} & -\frac{v \sinh p}{p} \\ 0 & \cosh p & \frac{v(\cosh p - 1)}{p} & -\sinh p \\ 0 & 0 & 1 & 0 \\ 0 & -\sinh p & \frac{v \sinh p}{p} & \cosh p \end{pmatrix}.$$

Let us find the the image of  $\Phi(\mathfrak{h})$  under the action of  $\Phi = Ad\xi$  on the one-dimensional subalgebra  $\mathfrak{h}$  with a direction vector  $e_4 + ye_2$ ,  $y \geq 0$ ;

$$\Phi(e_4 + ye_2) = (\cosh p - y \sinh p)e_4 + \frac{v}{p}[y(\cosh p - 1) - \sinh p]e_1 + (y \cosh p - \sinh p)e_2,$$

$$\Phi(e_4 + ye_2) = e_4 + \frac{v}{p}\left[\frac{y(\cosh p - 1) - \sinh p}{\cosh p - y \sinh p}\right]e_1 + \frac{y \cosh p - \sinh p}{\cosh p - y \sinh p}e_2,$$

$$x' = \frac{v}{p}\left[\frac{y(\cosh p - 1) - \sinh p}{\cosh p - y \sinh p}\right], \quad (5)$$

$$y' = +\frac{y \cosh p - \sinh p}{\cosh p - y \sinh p}.$$

By choosing  $p \neq 0$  such that  $y = \frac{\sinh p}{\cosh p - 1}$ , we obtain  $x' = 0$ . Let us note in addition that  $\coth p \neq 0$  that is the map is correctly defined. Applying to the obtain Bol algebra the automorphism of view 5, one can make  $y' = 1$ .

- If  $p = 0$  then  $x' = -v$   
 $y' = y$ ;
- if  $v \neq 0$  then one can make  $x' = 1, y' = 1$ ;
- if  $v = 0$  then  $x' = 0$  and, one can make  $y' = 1$ ;
- if  $y = 0$  then  $x' = -\frac{v}{p} \tanh p$ ,  
(6)  
 $y' = -\tanh p$ ,
- a) if  $p = 0$ , then  $x' = -v, v' = 0$

- b) if  $p \neq 0$  then when applying the automorphism of view (6) and with regards to  $v$  we can make  $x' = 0; 1, y' = 1$ .

The selected three cases are isotopic (in the sense of the definition of isotopy).

We note the exceptional Bol algebra of Theorem III.5.1 under the action of isotopic transformation is not changing

Summarizing the conducted examination one can formulate the Theorem:

**Theorem  $V^-$ .3.3** Any Bol algebra of dimension 3, with the trilinear operation of Type  $V^-$  and the canonical enveloping Lie algebra  $\mathfrak{G}$  of dimension 4, is isotopic to one of the following Bol algebras:

- $e_2 \cdot e_3 = -e_3, (e_2, e_3, e_2) = e_1, (e_2, e_3, e_3) = e_2;$
- $e_2 \cdot e_3 = -e_2, (e_2, e_3, e_2) = e_1, (e_2, e_3, e_3) = e_2;$
- $e_2 \cdot e_3 = -e_1 - e_2, (e_2, e_3, e_2) = e_1, (e_2, e_3, e_3) = e_2;$
- trivial bilinear operation,  $(e_2, e_3, e_2) = e_1, (e_2, e_3, e_3) = e_2.$

Analogically one can state the correctness of the Theorem.

**Theorem  $V^+$ .3.4** Any Bol algebra of dimension 3, with the trilinear operation of Type  $V^+$  and the canonical enveloping Lie algebra  $\mathfrak{G}$  of dimension 4, is isotopic to one of the following Bol algebras:

- $e_2 \cdot e_3 = -e_3, (e_2, e_3, e_2) = e_1, (e_2, e_3, e_3) = -e_2;$
- $e_2 \cdot e_3 = e_2, (e_2, e_3, e_2) = e_1, (e_2, e_3, e_3) = -e_2;$
- trivial bilinear operation,  $(e_2, e_3, e_2) = e_1, (e_2, e_3, e_3) = -e_2.$

Below we reduce to description of 3-Webs corresponding to the isolated Bol algebras of Type  $V^-$  and Type  $V^+$ .

The composition law  $(\Delta)$ , corresponding to the Lie group  $G$  of enveloping Lie algebra for Bol algebra is defined as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Delta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + \frac{x_4 y_2 - y_4 x_2}{2} \\ x_2 + y_2 \cos x_3 - y_4 \sin x_3 \\ x_3 + y_3 \\ x_4 - y_2 \sin(x_3) + y_4 \cos(x_3) \end{bmatrix}.$$

In case  $V^-.1$  the subgroup  $H = \exp \mathfrak{h}$ , can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{ \exp \alpha (e_4 + e_3) \}_{\alpha \in \mathbb{R}} = \{ 0, 0, \alpha, \alpha \}_{\alpha \in \mathbb{R}}.$$

The collection of elements

$$B = \exp \mathfrak{B} = \left\{ t + \frac{(v - \sin v)u^2}{2v^2}, \frac{u}{v} \sin v, v, \frac{u}{v}(1 - \cos v) \right\}_{t,u,v \in \mathbb{R}}$$

form a local section of left space coset  $G \bmod H$ .  $\exp : \mathfrak{G} \supset \mathfrak{B} \longrightarrow B \subset G$  and  $B = \exp \mathfrak{B}$

$$\exp^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - \frac{(x_2)^2 \sin^2(x_3)}{2(x_3)^3} + \frac{(x_2)^2 \sin^3(x_3)}{2(x_3)^4} \\ \frac{x_2}{x_3} \sin x_3 \\ x_3 \\ x_4 \end{bmatrix}$$

$x_1, x_2, x_3, x_4 \in \mathbb{R}$ .

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , in the neighborhood  $e$ , can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} x_1 + \frac{(x_3 - v)[x_2 + (x_3 - v) \sin v]}{2} \\ x_2 + (x_3 - v) \sin v \\ v \\ x_4 - (x_3 - v) \cos v \end{bmatrix} \triangle \begin{bmatrix} 0 \\ 0 \\ x_3 - v \\ x_3 - v \end{bmatrix},$$

where  $v$  are any numbers defined from the relation

$$[x_4 - (x_3 - v) \cos v] \sin v = [x_2 + (x_3 - v) \cos v] (\cos v - 1).$$

The composition law  $(\star)$  corresponding to the local analytical Bol loop  $B(\star)$  is defined as follows:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \triangle \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \right) \\ \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t + t' \\ u + u' \cos v \\ v + v' \\ u \sin(v) \end{bmatrix} \right) \right) \\ &= \exp^{-1} \left( \begin{bmatrix} t + t' + \frac{(v+v'-T)[u+u' \cos v + (v+v'-T) \sin T]}{2} \\ u + u' \cos v + (v + v' - T) \sin T \\ T \\ u \sin v - (v + v' - T) \cos T \end{bmatrix} \right) \\ &= \begin{bmatrix} F_1(t, t', u' u', v' v', T) \\ [u + u' \cos v + (v + v' - T) \sin T] \frac{\sin T}{T} \\ T \end{bmatrix}, \end{aligned}$$

where  $T$  is defined from the relation:

$$[u \sin v - (v + v' - T) \cos T] \sin T = [u + u' \cos v + (v + v' - T) \cos T] (\cos T - 1).$$

And  $F_1(t, t', u'u', v'v', T)$  from the relation

$$F_1(t, t', u'u', v'v', T) = t + t' + \frac{(v + v' - T)(u + u' \cos v)}{2} + \frac{(v + v' - T)^2}{2} \sin T - \frac{[u + u' \cos v + (v + v' - T) \sin T]^2}{2T^4} (T - \sin T) \sin^2 T. \quad (21)$$

In case  $V^-$  the subgroup  $H = \exp \mathfrak{h}$ , can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{ \exp \alpha(e_4 + ye_2) \}_{\alpha \in \mathbb{R}} = \{0, 0, \alpha, \alpha\}_{\alpha \in \mathbb{R}}.$$

The collection of elements

$$B = \exp \mathfrak{B} = \left\{ t + \frac{(v - \sin v)u^2}{2v^2}, \frac{u}{v} \sin v, v, \frac{u}{v}(1 - \cos v) \right\}_{t, u, v \in \mathbb{R}}$$

form a local section of left space coset  $G \bmod H$ .

Here  $\exp^{-1}$  is defined as in the case above.

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , in the neighborhood  $e$ , can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} x_1 AB \frac{y - y \cos x_3 - \sin x_3}{2x_3} \\ A \frac{\sin x_3}{x_3} \\ x_3 \\ A \frac{1 - \cos x_3}{x_3} \end{bmatrix} \Delta \begin{bmatrix} 0 \\ y \frac{x_4 \sin x_3 - x_2(1 - \cos x_3)}{y - y \cos x_3 + \sin x_3} \\ 0 \\ \frac{x_4 \sin x_3 - x_2(1 - \cos x_3)}{y - y \cos x_3 + \sin x_3} \end{bmatrix},$$

where  $A, B$  are any numbers defined from the relations

$$A = x_3 \frac{x_2(y \sin x_3 + \cos x_3) - x_4(y \cos x_3 - \sin x_3)}{y - y \cos x_3 + \sin x_3},$$

$$B = \frac{x_4 \sin x_3 - x_2(1 - \cos x_3)}{y - y \cos x_3 + \sin x_3}.$$

The composition law  $(\star)$  corresponding to the local analytical Bol loop  $B(\star)$  is defined as follows:

$$\begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} = \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \Delta \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \right)$$



$$\begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} = \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t+t' \\ u+u' \cos v \\ v+v' \\ -u \sin(v) \end{bmatrix} \right) \right) \\ = \begin{bmatrix} t+t'-C' \\ A' \frac{\sin^2(v+v')}{(v+v')^2} \\ v+v' \end{bmatrix},$$

where  $A', C'$  are defined from the relations:

$$A = (v+v') \frac{\{(u+u' \cos v) [(y \sin(v+v') + \cos(v+v')) - u \sin v [y \cos(v+v') - \sin(v+v')]]\}}{y - y \cos(v+v') + \sin(v+v')},$$

$$B = \frac{u \sin v \sin(v+v') - (u+u' \cos v)(1 - \cos(v+v'))}{y - y \cos(v+v') + \sin(v+v')},$$

$$C' = A'B' \frac{(y - y \cos(v+v') - \sin(v+v'))}{v+v'} + (A')^2 \frac{\sin^4(v+v')}{2(v+v')^4} \cdot \frac{-1 + \cos(v+v')}{v+v'}.$$

We pass to the description of Bol 3-Webs corresponding to the Type  $V^+$ .

In this case the composition law  $(\Delta)$ , corresponding to Lie group  $G$ , with enveloping Lie algebra of Bol algebra of Type  $V^+$  is defined as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Delta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + \frac{x_4 y_2 - y_4 x_2}{2} \\ x_2 + y_2 \cosh x_3 - y_4 \sinh x_3 \\ x_3 + y_3 \\ x_4 - y_2 \sinh(x_3) + y_4 \cosh(x_3) \end{bmatrix}.$$

In case  $V^+.1$  the subgroup  $H = \exp \mathfrak{h}$ , can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{\exp \alpha(e_4 + e_3)\}_{\alpha \in \mathbb{R}} = \{0, 0, \alpha, \alpha\}_{\alpha \in \mathbb{R}}.$$

The collection of elements

$$B = \exp \mathfrak{B} = \left\{ t + \frac{(v - \sinh v)u^2}{2v^2}, \frac{u}{v} \sinh v, v, \frac{u}{v}(1 - \cosh v) \right\}_{t, u, v \in \mathbb{R}}$$

form a local section of left space coset  $G \bmod H$ .  $\exp : \mathfrak{G} \supset \mathfrak{B} \longrightarrow B \subset G$  and  $B = \exp \mathfrak{B}$

$$\exp^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - \frac{(x_2)^2 \sinh^2(x_3)}{2(x_3)^3} + \frac{(x_2)^2 \sinh^3(x_3)}{2(x_3)^4} \\ \frac{x_2}{x_3} \sinh x_3 \\ x_3 \end{bmatrix}$$

$x_1, x_2, x_3, x_4 \in \mathbb{R}$ .

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , in the neighborhood  $e$ , can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} x_1 + \frac{(x_3-v)[x_2+(x_3-v)\sinh v]}{2} \\ x_2 + (x_3-v)\sin v \\ v \\ x_4 - (x_3-v)\cos v \end{bmatrix} \Delta \begin{bmatrix} 0 \\ 0 \\ x_3-v \\ x_3-v \end{bmatrix}$$

where  $v$  are any numbers defined from the relation

$$[x_4 - (x_3 - v) \cosh v] \sinh v = [x_2 + (x_3 - v) \cosh v] (\cosh v - 1).$$

The composition law  $(\star)$ , corresponding to the local analytical Bol loop  $B(\star)$ , is defined as follows:

$$\begin{aligned} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \Delta \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \right) \\ \begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} &= \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t+t' \\ u+u'\cosh v \\ v+v' \\ u\sinh(v) \end{bmatrix} \right) \right) \\ &= \exp^{-1} \left( \begin{bmatrix} t+t' + \frac{(v+v'-P)[u+u'\cosh v+(v+v'-P)\sinh P]}{2} \\ u+u'\cosh v + (v+v'-P)\sin P \\ P \\ u\sinh v - (v+v'-P)\cosh P \end{bmatrix} \right) \\ &= \begin{bmatrix} F_2(t, t', u'u', v'v', P) \\ [u+u'\cosh v + (v+v'-P)\sinh P] \frac{\sinh P}{P} \\ P \end{bmatrix} \end{aligned}$$

where  $P$  is defined from the relation:

$$[u\sinh v - (v+v'-P)\cosh P] \sinh P = [u+u'\cosh v + (v+v'-P)\cosh P] (\cosh P - 1)$$

and  $F_1(t, t', u'u', v'v', P)_1$  from the relation

$$\begin{aligned} F_1(t, t', u'u', v'v', P) &= t+t' + \frac{(v+v'-P)(u+u'\cosh v)}{2} + \frac{(v+v'-P)^2}{2} \sinh P - \\ &\quad - \frac{[u+u'\cosh v + (v+v'-P)\sinh P]^2}{2P^4} (P - \sinh P) \sinh^2 P. \quad (22) \end{aligned}$$

In case  $V^+$  the subgroup  $H = \exp \mathfrak{h}$ , can be realized as the collection of elements

$$H = \exp \mathfrak{h} = \{\exp \alpha(e_4 + ye_2)\}_{\alpha \in \mathbb{R}} = \{0, 0, \alpha, \alpha\}_{\alpha \in \mathbb{R}}.$$

The collection of elements

$$B = \exp \mathfrak{B} = \left\{ t + \frac{(v - \sinh v)u^2}{2v^2}, \frac{u}{v} \sinh v, v, \frac{u}{v}(1 - \cosh v) \right\}_{t, u, v \in \mathbb{R}}$$

form a local section of left space coset  $G \text{ mod } H$ .

Here  $\exp^{-1}$  is defined as in the case above.

Any element  $(x_1, x_2, x_3, x_4)$  from  $G$ , in the neighborhood  $e$ , can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} x_1 D E \frac{y - y \cosh x_3 - \sinh x_3}{2x_3} \\ D \frac{\sinh x_3}{x_3} \\ x_3 \\ D \frac{1 - \cosh x_3}{x_3} \end{bmatrix} \triangle \begin{bmatrix} 0 \\ y \frac{x_4 \sinh x_3 - x_2(1 - \cosh x_3)}{y - y \cosh x_3 + \sinh x_3} \\ 0 \\ \frac{x_4 \sinh x_3 - x_2(1 - \cosh x_3)}{y - y \cosh x_3 + \sinh x_3} \end{bmatrix},$$

where  $D, E$  are any numbers defined from the relations

$$D = x_3 \frac{[x_2(y \sinh x_3 + \cosh x_3) - x_4(y \cosh x_3 - \sinh x_3)]}{y - y \cosh x_3 + \sinh x_3},$$

$$E = \frac{x_4 \sinh x_3 - x_2(1 - \cosh x_3)}{y - y \cosh x_3 + \sinh x_3}.$$

The composition law  $(\star)$  corresponding to the local analytical Bol loop  $B(\star)$  is defined as follows:

$$\begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} = \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t \\ u \\ v \\ 0 \end{bmatrix} \triangle \begin{bmatrix} t' \\ u' \\ v' \\ 0 \end{bmatrix} \right) \right)$$

$$\begin{pmatrix} t \\ u \\ v \end{pmatrix} \star \begin{pmatrix} t' \\ u' \\ v' \end{pmatrix} = \exp^{-1} \left( \prod_B \left( \begin{bmatrix} t + t' \\ u + u' \cosh v \\ v + v' \\ -u \sinh(v) \end{bmatrix} \right) \right)$$

$$= \begin{bmatrix} t + t' - F' \\ D' \frac{\sinh^2(v+v')}{(v+v')^2} \\ v + v' \end{bmatrix}$$

where  $D', F'$  are defined from the relations:

$$D' = (v + v') \frac{\Lambda}{y - y \cosh(v + v') + \sinh(v + v')},$$

$$\Lambda = \{(u + u' \cosh v) [(y \sinh(v + v') + \cosh(v + v')) - u \sinh v [y \cosh(v + v') - \sinh(v + v')]]\}, \quad (23)$$

$$E' = \frac{u \sinh v \sinh(v + v') - (u + u' \cosh v)(1 - \cosh(v + v'))}{y - y \cosh(v + v') + \sinh(v + v')},$$

$$F' = D' E' \frac{(y - y \cosh(v + v') - \sinh(v + v'))}{v + v'} + (A')^2 \frac{\sinh^4(v + v')}{2(v + v')^4} \cdot \frac{-1 + \cosh(v + v')}{v + v'}.$$

## 4.6 BOL ALGEBRAS WITH TRILINEAR OPERATION OF TYPE VI

As in the previous chapter, we will base our investigation of 3-dimensional Bol algebras, on the examination of their canonical enveloping Lie algebras. In what follows, we consider Bol algebras of dimension 3, from their construction see [30,34]; it follows that the dimension of their canonical enveloping Lie algebras, can not be more than 6. Below we limit ourselves to the classification of Bol algebras (and their corresponding 3-webs), with canonical enveloping Lie algebras of dimension  $\leq 5$ .

Let  $\mathfrak{B}$  be a 3-dimensional Bol algebra with a trilinear operation Type VI see chapter II §3, and  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}$ -its canonical enveloping Lie algebra according to the table given in chapter II (case 4). We note that the situation  $\dim \mathfrak{G} = 3$  is possible, that means we obtain a total grouped 3-Web the corresponding Lie group  $G$ , is isomorphic to the matrix of the view:

$$\begin{pmatrix} \cosh x & -\sinh x & \frac{z(1-\cosh x)+y \sinh x}{x} \\ \sinh x & -\cosh x & \frac{z(1-\cosh x)+z \sinh x}{x} \\ 0 & 0 & 1 \end{pmatrix},$$

with  $x, y, z \in \mathbb{R}$ .

Let us examine the case  $\dim \mathfrak{G} = 4$ , the structural constants of Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$ ,  $\mathfrak{B} = \langle e_1, e_2, e_3 \rangle$  are defined as follows:

$$[e_2, e_3] = e_4, \quad [e_3, e_4] = -e_1, \quad (1)$$

$$[e_1, e_3] = -e_5, \quad [e_3, e_5] = -e_2,$$

in addition  $\mathfrak{G} = \mathfrak{B} \dot{+} [\mathfrak{B}, \mathfrak{B}]$ ,  $[\mathfrak{B}, \mathfrak{B}] = \langle e_4 \rangle$ .

By introducing in consideration the 3-dimensional subspaces of subalgebras

$$\mathfrak{h}_{x,y,z,x_1,y_1,z_1} = \langle e_4 + xe_1 + ye_2 + ze_3, e_5 + x_1e_1 + y_1e_2 + z_1e_3 \rangle,$$

where  $x, y, z, x_1, y_1, z_1 \in \mathbb{R}$  Let us note

$$e'_4 = e_4 + xe_1 + ye_2 + ze_3,$$

$$e'_5 = e_5 + x_1e_1 + y_1e_2 + z_1e_3. \quad (2)$$

We are interested in those spaces which are Lie subalgebras in  $\mathfrak{G}$ , that is  $[e'_4, e'_5] \in \langle e'_4, e'_5 \rangle$ .

The condition of algebraical closure of the subspaces  $\mathfrak{h}_{x,y,z,x_1,y_1,z_1}$  consists that there exist  $\alpha, \beta \in \mathbb{R}$  such that:

$$[e'_4, e'_5] = \alpha e'_4 + \beta e'_5,$$

that is

$$\beta = x_1 z - z_1 x, \alpha = y z_1 - z y_1, \quad (*)$$

$$z_1 = \alpha x + \beta x_1, \alpha z + \beta z_1 = 0, \quad (3)$$

$$-z = \alpha y + \beta y_1.$$

1. 1. If  $\beta = 0$  then,  $\alpha \neq 0$  or  $\alpha = 0$ .

- If  $\alpha \neq 0$ , then  $z = y = 0$  and we come to the contradiction with the condition (\*).
- $\alpha = 0$ , then  $z = z_1 = 0$  and, the subspace will be defined as follows:

$$\langle e_4 + x e_1 + y e_2, e_5 + x_1 e_1 + y_1 e_2 \rangle,$$

it is a subalgebra we note it  $\mathfrak{h}_{x,y,0,x_1,y_1,0}$ .

2. 2. If  $\beta \neq 0$  assume that:

- 2.1.  $x_1 = y_1 = 0, z_1 = t > 0$ , then  $\beta = tx, \alpha = yt$ . And, associating it with (3), we obtain:

$$\begin{cases} t = tyx, \\ -z = ty^2 \\ -yt - xt^2 \end{cases}$$

then  $xy = 1$ , hence  $x \neq 0, y \neq 0, z \neq 0$ ;

$$\begin{cases} yx = 1, \\ x = -y^3 \end{cases}$$

, hence  $y^4 = -1$ , and we obtain a contradiction

- 2.2  $x = y = 0$ , thus  $\beta = tx_1, \alpha = -iy_1, z = p > 0, z_1 = px_1^2$ ,  $-p = \beta y_1 = x_1 y_1 p$  hence,

$$\begin{cases} y_1 \cdot x_1 = -1, \\ (x_1)^4 = -1 \end{cases}$$

and we also come to the contradiction.

- 2.3  $x_1 = z = 0$ , then  $\beta = -xz_1, \alpha = yz_1, \beta z_1 = 0$ , hence  $z = 0$  we obtain a contradiction.
- 2.4  $x = z_1 = 0$ , then  $\beta = -zx_1, \alpha = -zy_1, \alpha z = 0$ .
  - 2.4.1 If  $\alpha = 0$  then  $z = 0$ , and come to a contradiction.
  - 2.4.2 If  $\alpha = 0$ , then  $-zy_1 = 0$ , hence  $y_1 = 0$  hence,  $z_1 = 0$  also we obtain a contradiction.
- 2.5  $y = y_1 = 0$ , then  $\alpha = z = 0, \beta = -xz_1, \beta z_1 = 0$ , hence,  $z_1 = 0$  also a contradiction.

- 2.6  $y = z = 0$ , then  $\beta = zx_1, \alpha = -zy_1, \beta z_1 = 0$ , hence,  $x_1 = 0$  we still get a contradiction.
- 2.7  $y = z = 0$ , then  $\alpha z = 0$ ; here  $z \neq 0$ , otherwise it will contradict the condition  $\beta \neq 0$ ; hence  $\alpha = 0, \beta = zx_1, \alpha y = -z = 0$  this is also a contradiction.
- 2.8  $y_1 = z = 0$ , then  $z_1 = 0$  it's a contradiction.
- 2.9  $y = z_1 = 0, \alpha = -zy_1, \beta = zx_1, z \neq 0, \alpha z = 0$ , hence,  $\alpha = 0, y_1 = 0$ , this implies  $z = 0$  it's a contradiction.

Summarizing: the condition  $\beta \neq 0$  its impossible that is  $\beta = \alpha = 0$ .

The analogical reasoning can be done if we consider  $\alpha = 0$ . In result we obtain only the subalgebra:

$$\langle e_4 + xe_1 + ye_2, e_5 + x_1e_1 + y_1e_2 \rangle.$$

The group of automorphisms  $F$  of Lie triple system  $\mathfrak{B}$  relatively to a fixed base  $e_1, e_2, e_3$  from Type  $IV^-$  is defined as follows:

$$F = \left\{ P = \begin{pmatrix} \alpha & -\beta & d \\ \beta & \alpha & f \\ 0 & 0 & \pm 1 \end{pmatrix}, \alpha^2 + \beta^2 \neq 0, \alpha, \beta, d, f \in \mathbb{R} \right\}.$$

The extension of automorphism  $P$  from  $F$  to the automorphism of Lie algebra  $\mathfrak{G}$ , transforming the subspace  $\mathfrak{B}$  into itself can be realized as follows:

$$Ae_4 = [Ae_2, Ae_3] = \pm\beta e_5 \pm \alpha e_4,$$

$$Ae_5 = [Ae_1, Ae_3] = \pm\alpha e_5 \pm \beta e_4.$$

In addition for the subspace  $\mathfrak{h}_{x,y,0,x_1,y_1,0}$  we have:

$$A(e_4 + xe_1 + ye_2) = e_4 \pm \frac{x\alpha^2 + y\beta^2 - y\alpha\beta - \beta\alpha x}{\alpha^2 + \beta^2} e_1 \pm \frac{y\alpha^2 - x\beta^2 + x\alpha\beta - y\alpha\beta}{\alpha^2 + \beta^2} e_2, \quad (4)$$

$$A(e_5 + x_1e_1 + y_1e_2) = e_5 \pm \frac{x_1\alpha^2 + y_1\beta^2 - y_1\alpha\beta + \beta\alpha x_1}{\alpha^2 + \beta^2} e_1 \pm \frac{y_1\alpha^2 + x_1\beta^2 + x\alpha\beta + y_1\alpha\beta}{\alpha^2 + \beta^2} e_2,$$

such that the action of  $A$  on the subspace  $\mathfrak{h}_{x,y,0,x_1,y_1,0}$ , can be represented as follows:

$$\begin{pmatrix} \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} & -\frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \\ \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} & \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \end{pmatrix} \begin{pmatrix} x & x_1 \\ y & y_1 \end{pmatrix} \begin{pmatrix} \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} & \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \\ -\frac{\beta}{\sqrt{\alpha^2 + \beta^2}} & \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \end{pmatrix}.$$

If the matrix

$$\begin{pmatrix} \lambda_1 & \delta \\ 0 & \lambda_2 \end{pmatrix}$$

, with  $\lambda_1, \delta \in \mathbb{R}$ . (6)

If the matrix

$$\begin{pmatrix} x & x_1 \\ y & y_1 \end{pmatrix},$$

has two complex conjugate eigenvalues  $\lambda_1 = \alpha + i\beta$  ( $\lambda_2 = \alpha - i\beta$ ) correspondently, where  $\beta \neq 0$ , and eigenvectors  $\bar{a} = \bar{\xi} + i\bar{\eta}$  ( $\bar{a} = \bar{\xi} - i\bar{\eta}$ ) correspondently.

Let  $\eta \neq 0$  the first vector base  $(e_1, e_2) \longrightarrow (\bar{\eta}, \bar{\epsilon})$ , where  $\bar{\epsilon}$  is perpendicular to  $\bar{\eta}$

$$A\bar{a} = A\bar{\xi} + iA\bar{\eta} = (\alpha\bar{\xi} - \beta\bar{\eta}) + i(\alpha\bar{\eta} + \beta\bar{\xi}),$$

$$A\bar{\eta} = \alpha\bar{\eta} + \beta\bar{\xi}, \quad (7),$$

where  $\bar{\xi} = \mu\bar{\eta} + \nu\bar{\epsilon}$

$$A\bar{\xi} = \alpha\bar{\xi} - \beta\bar{\eta} = \mu A\bar{\eta} + \nu A\bar{\epsilon}.$$

Thus we can write out  $A\bar{\eta} = (\alpha + \beta\mu)\bar{\eta} + \beta\mu\bar{\epsilon}$ ,  
on the other hand

$$\nu A\bar{\epsilon} = -\mu A\bar{\eta} + \alpha\bar{\xi} - \beta\bar{\eta} = \bar{\eta}(-\beta - \mu^2\beta) + \bar{\epsilon}(-\mu\beta\nu + \alpha\nu), \quad \nu \neq 0.$$

Hence the canonical view of the matrix will be represented as follows:

$$\begin{pmatrix} \alpha + \beta\mu & -\frac{\beta}{\nu}(1 + \mu^2) \\ \beta\nu & \alpha - \mu\beta \end{pmatrix} \nu \neq 0, \beta \neq 0. \quad (8)$$

Alternative approach: if eigenvalues  $\lambda_1 = \alpha + i\beta$  ( $\lambda_2 = \alpha - i\beta$ ) are two complex conjugate numbers corresponding to two complex eigenvectors  $\bar{a} = \bar{\xi} + i\bar{\eta}$  ( $\bar{b} = \bar{\xi} - i\bar{\eta}$ ). Let's consider vectors  $\bar{\nu}, \bar{\mu}$  defined in terms of  $\bar{a}$  and  $\bar{b}$  as follow:

$$\bar{\nu} = \frac{\bar{a} + \bar{b}}{2}, \bar{\nu} = \frac{\bar{a} - \bar{b}}{2i}. \quad (9)$$

The are reals, moreover we have:

$$A\bar{\nu} = \alpha\bar{\nu} - \beta\bar{\mu}$$

$$(10)$$

$$A\bar{\mu} = \beta\bar{\nu} + \alpha\bar{\mu}$$

Therefore the linear span in the space of reals numbers  $\mathbb{R}$ , constructed in terms of vectors in (9), is an invariant subspace under the action of automorphism  $A$ . Therefore the matrix induced by the automorphism in that subspace, in the base defined in (9) is:



$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \beta \neq 0, \alpha \in \mathbb{R}.$$

We then obtain the following theorem:

**Theorem III.6.1** Any Bol algebra of dimension 3, with the trilinear operation of Type  $III^-$ , and the canonical enveloping Lie algebra of dimension 5, is isomorphic to one of Bol algebras below:

- $VI.1.e_2 \cdot e_3 = -\lambda_1 e_1, (e_2, e_3, e_3) = e_1,$
- $e_1 \cdot e_3 = -\lambda_2 e_1 - \delta e_2, (e_3, e_1, e_3) = e_2, \lambda_1 \neq 0, \lambda_2 \neq 0, \delta \in \mathbb{R}.$
- $VI.2.e_2 \cdot e_3 = -(\alpha + \beta\mu)e_1 - \beta\nu e_2, (e_2, e_3, e_3) = e_1, \nu > 0, \beta > 0,$
- $e_1 \cdot e_3 = \frac{\beta(1+\mu^2)}{\nu}e_1 - (\alpha - \mu)e_2, (e_3, e_1, e_3) = e_2, \mu, \alpha \geq 0, \nu \neq 0,$   
or  
;  $e_2 \cdot e_3 = -\alpha e_1 + \beta e_2, e_1 \cdot e_3 = -\beta e_1 - \alpha e_2, \alpha \geq 0, \beta > 0.$

We note that the distinguished Bol algebras are not isomorphic among themselves.

The composition law ( $\triangle$ ), of local Lie group  $G$ , is defined as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \triangle \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} F_1(x_1, x_3, y_1, y_2, y_4, y_5) \\ F_2(x_2, x_3, y_1, y_2, y_4, y_5) \\ x_3 + y_3 \\ F_4(x_4, x_3, y_1, y_2, y_4, y_5) \\ F_5(x_5, x_3, y_1, y_2, y_4, y_5) \end{bmatrix}.$$

Where

$$\begin{aligned} F_1(x_1, x_3, y_1, y_2, y_4, y_5) &= x_1 + (\cosh(x_3 \frac{\sqrt{2}}{2}) \cos(x_3 \frac{\sqrt{2}}{2}))y_1 + (\sinh(x_3 \frac{\sqrt{2}}{2}) \sin(x_3 \frac{\sqrt{2}}{2}))y_2 \\ &- \frac{\sqrt{2}}{2} \left[ \sinh(x_3 \frac{\sqrt{2}}{2}) \cos(x_3 \frac{\sqrt{2}}{2}) + \cosh(x_3 \frac{\sqrt{2}}{2}) \sin(x_3 \frac{\sqrt{2}}{2}) \right] y_4 \\ &- \frac{\sqrt{2}}{2} \left[ \sinh(x_3 \frac{\sqrt{2}}{2}) \cos(x_3 \frac{\sqrt{2}}{2}) + \cosh(x_3 \frac{\sqrt{2}}{2}) \sin(x_3 \frac{\sqrt{2}}{2}) \right] y_5, \quad (24) \end{aligned}$$

$$\begin{aligned} F_2(x_2, x_3, y_1, y_2, y_4, y_5) &= x_2 + \frac{\sqrt{2}}{2} [\cos(x_3 \frac{\sqrt{2}}{2}) \sinh(x_3 \frac{\sqrt{2}}{2}) - \sin(x_3 \frac{\sqrt{2}}{2}) \cosh(x_3 \frac{\sqrt{2}}{2})] y_1 \\ &+ (\cosh(x_3 \frac{\sqrt{2}}{2}) \cos(x_3 \frac{\sqrt{2}}{2}))y_2 - (\sinh(x_3 \frac{\sqrt{2}}{2}) \sin(x_3 \frac{\sqrt{2}}{2}))y_4 \\ &+ \frac{\sqrt{2}}{2} [\cosh(x_3 \frac{\sqrt{2}}{2}) \sin(x_3 \frac{\sqrt{2}}{2}) + \sinh(x_3 \frac{\sqrt{2}}{2}) \cos(x_3 \frac{\sqrt{2}}{2})] y_5, \quad (25) \end{aligned}$$

$$\begin{aligned}
F_4(x_4, x_3, y_1, y_2, y_4, y_5) = & x_4 - (\sinh(x_3 \frac{\sqrt{2}}{2}) \sin(x_3 \frac{\sqrt{2}}{2})) y_1 \\
& - \frac{\sqrt{2}}{2} \left[ \sinh(x_3 \frac{\sqrt{2}}{2}) \cos(x_3 \frac{\sqrt{2}}{2}) + \cosh(x_3 \frac{\sqrt{2}}{2}) \sin(x_3 \frac{\sqrt{2}}{2}) \right] y_2 + \\
& + (\cosh(x_3 \frac{\sqrt{2}}{2}) \cos(x_3 \frac{\sqrt{2}}{2})) y_4 - \\
& - \frac{\sqrt{2}}{2} \left[ \sinh(x_3 \frac{\sqrt{2}}{2}) \cos(x_3 \frac{\sqrt{2}}{2}) + \cosh(x_3 \frac{\sqrt{2}}{2}) \sin(x_3 \frac{\sqrt{2}}{2}) \right] y_5, \quad (26)
\end{aligned}$$

$$\begin{aligned}
F_5(x_5, x_3, y_1, y_2, y_4, y_5) = & x_5 + \frac{\sqrt{2}}{2} \left[ \sinh(x_3 \frac{\sqrt{2}}{2}) \cos(x_3 \frac{\sqrt{2}}{2}) + \sin(x_3 \frac{\sqrt{2}}{2}) \cosh(x_3 \frac{\sqrt{2}}{2}) \right] y_1 + \\
& + (\sinh(x_3 \frac{\sqrt{2}}{2}) \sin(x_3 \frac{\sqrt{2}}{2})) y_2 - \\
& - \frac{\sqrt{2}}{2} \left[ \sinh(x_3 \frac{\sqrt{2}}{2}) \cos(x_3 \frac{\sqrt{2}}{2}) + \sin(x_3 \frac{\sqrt{2}}{2}) \cosh(x_3 \frac{\sqrt{2}}{2}) \right] y_4 + \\
& + (\cosh(x_3 \frac{\sqrt{2}}{2}) \cos(x_3 \frac{\sqrt{2}}{2})) y_5 \quad (27)
\end{aligned}$$

For this particular type of Bol algebras the classification with accuracy to isotopy and the description of the corresponding 3-Webs are not given for reason of awkwardness.

## 4.7 BOL ALGEBRAS WITH TRILINEAR OPERATION OF TYPE VII

As in the last chapter we will base our investigation of 3-dimensional Bol algebras on the examination of their canonical enveloping Lie algebras. Here we limit ourselves to the classification of Bol algebras (and their corresponding 3-webs), with canonical enveloping Lie algebras of dimension  $\leq 5$ .

Let  $\mathfrak{B}$  be a 3-dimensional Bol algebra with a trilinear operation Type VII see chapter II §3, and  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}$ -its canonical enveloping Lie algebra, according to the table given in chapter II (case 4). We note that the situation  $\dim \mathfrak{G} = 3$  is impossible that mean we obtain a total grouped 3-Web is excluded.

The structural constants of Lie algebra  $\mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle$ ,  $\mathfrak{B} = \langle e_1, e_2, e_3 \rangle$  are defined as follows:

$$[e_2, e_3] = e_4, \quad [e_1, e_4] = -e_1,$$

$$[e_1, e_3] = -e_5, \quad [e_2, e_5] = e_1, \quad [e_4, e_5] = e_5,$$

in addition  $\mathfrak{G} = \mathfrak{B} \dot{+} [\mathfrak{B}, \mathfrak{B}]$ ,  $[\mathfrak{B}, \mathfrak{B}] = \langle e_4 \rangle$ .

By introducing in consideration the 3-dimensional subspaces of subalgebras  $\mathfrak{h}_{x,y,z,x',y',z'}$  such that:

$$\mathfrak{h}_{x,y,z,x',y',z'} = \langle e_4 + xe_1 + ye_2 + ze_3, e_5 + x'e_1 + y'e_2 + z'e_3 \rangle_{x,y,z,x',y',z' \in \mathbb{R}}.$$

Let us note

$$e'_4 = e_4 + xe_1 + ye_2 + ze_3,$$

$$e'_5 = e_5 + x_1e_1 + y_1e_2 + z_1e_3.$$

Our interest goes to those spaces which are Lie subalgebras in  $\mathfrak{G}$ , that is  $[e'_4, e'_5] \in \langle e'_4, e'_5 \rangle$ .

where

$$[e'_4, e'_5] = (x' + y)e_1 + (yz' - zy')e_4 + (1 + xz' - zy')e_5,$$

so that the matrix

$$\begin{pmatrix} x & y & z & 1 & 0 \\ x' & y' & z' & 0 & 1 \\ x' + y & 0 & 0 & yz' - zy' & 1 + xz' - zy' \end{pmatrix},$$

must be linearly dependent or what equivalent to:

1. (1)  $(x' + y)(yz' - y'z) = 0$ ,
2. (2)  $(yz' - y'z)^2 - y'(x' + y) = 0$ ,

3. (3)  $y'(x' + y) + (1 + xz' - zx')(xy' - x'y) = 0$ ,
4. (4)  $(yz' - y'z)^2 = 0$ ,
5. (5)  $(yz' - y'z)(1 + xz' - zx') = 0$ ,
6. (6)  $(yz' - y'z)(xz' - x'z) - z'(x' + y) = 0$ ,
7. (7)  $z(x' + y) + (1 + xz' - zx')(xz' - x'z) = 0$ ,
8. (8)  $(x' + y) - x'(1 + xz' - zx') - x(yz' - y'z) = 0$ ,
9. (9)  $y(yz' - zy') - y'(1 + xz' - zx') = 0$ ,
10. (10)  $z(yz' - zy') - z'(1 + xz' - zx') = 0$ .

The system of relation (1)–(10) equivalent to the following:

- (2)'  $y'(x' + y) = 0$ ,
  - (3)'  $y(x' + y) + (1 + xz' - zx')(xy' - x'y) = 0$ ,
  - (6)'  $z'(x' + y) = 0$ ,
  - (8)'  $(x' + y) - x'(1 + xz' - zx') = 0$ ,
  - (9)'  $y'(1 + xz' - zx') = 0$ ,
  - (10)'  $z'(1 + xz' - zx') = 0$ ,
1. I. case 1. Let  $x' + y \neq 0$ , then  $y' = z' = 0$  and  $y = -(x')^2 z$ . in result we obtain the family of subalgebras  $\mathfrak{h}$ :

$$\mathfrak{h} = \langle e_4 + xe_1 - (x')^2 ze_2, e_5 + x'e_1 \rangle .$$

2. II case 2. Let  $x' + y = 0$ , then

- (3)'  $(1 + xz' + zy')(xy' + y^2) = 0$ ,
  - (7)'  $(1 + xz + zy)(xz' + yz) = 0$ ,
  - (8)'  $y(1 + xz' + zy) = 0$ ,
  - (9)'  $y'(1 + xz' + zy) = 0$ ,
  - (10)'  $z'(1 + xz' + zy) = 0$ ;
- a) if  $1 + xz' + zy = 0$  then

$$-x' = y, \quad yz' - zy' = 0,$$

and we obtain the family of subalgebras  $\mathfrak{h}$

$$\mathfrak{h} = \langle e_4 + xe_1 + ye_2 + ze_3, e_5 + ye_1 + y'e_2 + z'e_3 \rangle$$

where  $1 + xz' + zy = 0$  and  $yz' - zy' = 0$

- b) if  $1 + xz' + zy \neq 0$ , then  $x' = y' = z' = y = 0$ , and we obtain the family of subalgebras  $\mathfrak{h}$ :

$$\mathfrak{h} = \langle e_4 + xe_1 + ze_3, e_5 \rangle.$$

In result we obtain the collection of Bol algebras with structural equations given as follows:

1.  $e_2 \cdot e_3 = -xe_1 + (x')^2ze_2 - ze_3, (e_2, e_3, e_1) = e_1;$   
 $e_1 \cdot e_3 = -x'e_1, (e_3, e_1, e_2) = e_1,$
2.  $e_2 \cdot e_3 = -xe_1 - ye_2 - ze_3, (e_2, e_3, e_1) = e_1;$   
 $e_1 \cdot e_3 = ye_1 - y'e_2 - z'e_3, (e_3, e_1, e_2) = e_1,$  with condition

$$1 + xz' + zy = 0, yz' - zy' = 0, \quad (11)$$

3.  $e_2 \cdot e_3 = -xe_1 - ze_3, (e_2, e_3, e_1) = e_1;$   
 $(e_3, e_1, e_2) = e_1.$

The group of automorphisms  $F$ , of Lie triple system  $\mathfrak{B}$ , relatively to a fixed base  $e_1, e_2, e_3$  is defined as follows:

$$P = \left\{ P = \begin{pmatrix} \alpha & a & d \\ 0 & b & f \\ 0 & 0 & g \end{pmatrix}, \alpha \neq 0, bg = 1bd = af \right\}.$$

The extension of automorphism  $P$  from  $F$  to the automorphism of Lie algebra  $\mathfrak{G}$ , transforming the subspace  $\mathfrak{B}$  into itself can be realized as follows:

$$Pe_4 = [Pe_2, Pe_3] = e_4 - age_5,$$

$$Pe_5 = [Pe_1, Pe_3] = g\alpha e_5.$$

In addition the examination of Lie subalgebras in (11) gives: for 1.

$$P(e_4 + xe_1 - (x')^2ze_2 + ze_3) = e_4 + (x\alpha - (x')^2za + zd + ax')e_1 + (zf - (x')^2zb)e_2 + gze_3,$$

$$P(e_5 + x'e_1) = e_5 + \frac{x'}{g}e_1.$$

We will denote:

$$x_1 = x\alpha - (x')^2za + zd + ax'$$

$$y_1 = -(x')^2zb + zf,$$

$$z_1 = gz,$$

$$x'_1 = \frac{x'}{g},$$

$$y'_1 = z'_1 = 0.$$

1. I. If  $x' \neq 0$ , by acting on  $g$  we can make  $x'_1 = 1$  and here two cases must be investigate:
  - I.a) if  $z = 0$ , then  $z_1 = y_1 = 0$ . By acting on  $a$  and  $\alpha$  we can make  $x_1 = 0$ ;
  - I.b) if  $z \neq 0$  then  $z_1 = \beta \neq 0$ , hence by acting on  $a, f$  and  $\alpha$  we can make  $y_1 = x_1 = 0$ .
2. II. If  $x' = 0$  then  $x'_1 = 0$ ;
  - II.a) if  $z = 0$ , then  $z_1 = y_1 = 0$ . By acting on  $\alpha$  and  $a$  we can make  $x_1 = 0$
  - II.b) if  $z \neq 0$ , then by acting on  $g$  we can make  $z_1 = 1$ , while acting on  $a, f$  and  $\alpha$  we make  $y_1 = x_1 = 0$ .

In result we obtain for  $(x, y, z, x', y', z')$  one family of values and 3 exceptional six-uplet values:

$$\begin{aligned}
 &(0, 0, 0, 1, 0, 0), \\
 &(0, 0, \beta, 1, 0, 0), \quad \beta \neq 0 \\
 &(0, 0, 0, 0, 0, 0), \\
 &(0, 0, 1, 0, 0, 0),
 \end{aligned}$$

Now we pass to the examination of Bol algebras from the family of Lie subalgebras 2 in (11).

$$\begin{aligned}
 P(e_4 + xe_1 + ye_2 + ze_3) = e_4 + &\left[ (x + x' \frac{a}{\alpha})\alpha + (y + y' \frac{a}{\alpha})a + (z + z' \frac{a}{\alpha})d \right] e_1 + \\
 &+ \left[ (y + y' \frac{a}{\alpha})b + (z + z' \frac{a}{\alpha})f \right] e_2 + (z + z' \frac{a}{\alpha})ge_3, \quad (28)
 \end{aligned}$$

$$P(e_5 - ye_1 + y'e_2 + z'e_3) = e_5 + \frac{-y\alpha + y'a + z'd}{g\alpha}e_1 + \frac{y'b + z'f}{g\alpha}e_2 + \frac{z'}{\alpha}e_3,$$

where we will denote

$$\begin{aligned}
 x_1 &= (x\alpha - ya) + (y + y' \frac{a}{\alpha})a + (z + z' \frac{a}{\alpha})d, \\
 y_1 &= (y + y' \frac{a}{\alpha})b + (z + z' \frac{a}{\alpha})f, \\
 z_1 &= (z + z' \frac{a}{\alpha})g, \\
 x'_1 &= \frac{-y\alpha + y'a + z'd}{g\alpha}, \\
 y'_1 &= \frac{y'b + z'f}{g\alpha},
 \end{aligned}$$

$$z'_1 = \frac{z'}{\alpha},$$

with the conditions

$$1 + xz' + zy = 0 \quad (14)$$

$$yz' - zy' = 0.$$

1. I. If  $z' = 0$ , then  $z'_1 = 0$  and from (14) follow that,  $zy = -1$  and make  $zy' = 0$  then  $y' = 0, z \neq 0$  hence  $y'_1 = 0$ , by acting on  $g$  we can make  $x'_1 = 1, z_1 = \omega \neq 0$ , while acting on  $a, f$  and  $\alpha$  we can make  $x_1 = y_1 = 0$ .
2. II. If  $z' \neq 0$ , then by acting on  $\alpha$  we can in additional  $z'_1 = 1$ 
  - II.a) if  $y' = 0$ , then from (14) it follow that,  $yz' = 0$  hence  $y = 0$  therefore  $xz' = -1$ ; hence acting on  $g, a, f$  we can make  $y'_1 = 0; 1, x'_1 = \lambda \neq 0, z_1 = 0, y_1 = 0, x_1 = s$  (where  $s$  any number)
  - II.b) if  $y' \neq 0$  and  $z = 0$  from (14) it follow that:  $xz' = -1, yz' = 0$ , hence  $y = 0$ , then acting on  $f$  we can make  $y'_1 = 0$  and  $x'_1 = t$  (where  $t$  is any number) acting on  $g$  we can make  $z_1 = 1, y_1 = 0, x_1 = \gamma$  - (any number).
  - II.b.2) If  $y' \neq 0$  and  $z \neq 0$ , then by acting on  $f$  and  $b$  we make  $y'_1 = 0, x'_1 = \tau$  - any number,  $z_1 = \kappa \neq 1, y_1 = 0, x_1 = \eta$  -any number.

In result we obtain for  $(x, y, z, x', y'z')$  six families of values:

$$(0, 0, \omega, 1, 0, 0)\omega \neq 0,$$

$$(s, 0, 0, \lambda, 1, 1), \forall s$$

$$(s, 0, 0, \lambda, 0, 1), \lambda \neq 0, \forall s$$

$$(\gamma, 0, 1, t, 0, 1), \forall \gamma, t$$

$$(\eta, 0, \kappa, \tau, 0, 1), \kappa \neq 1.$$

We pass to the examination of Bol algebras from the family 3 of (11)

$$P(e_4 + xe_1 + xe_3) = e_4 + (x\alpha + zd)e_1 + zfe_2 + zye_3,$$

$$P(e_5) = e_5,$$

where we can denote

$$x_1 = x\alpha + zd,$$

$$y_1 = zf,$$

$$z_1 = zy,$$

$$x'_1 = y'_1 = z'_1 = 0.$$

- a) If  $z = 0$ , then  $z_1 = y_1 = 0$ , therefore  $x_1 = x\alpha$ .
  - a.1. If  $x \neq 0$ , by choosing  $\alpha = \frac{1}{x}$  we can make  $x_1 = 1$ .
  - a.2. If  $x = 0$ , then  $z_1 = y_1 = x_1 = 0$ .
- b) If  $z \neq 0$ , by choosing  $g = \frac{1}{z}$  we can make  $z_1 = 1, z = b$ , but if  $f = 0$  we make  $y_1 = 0$  by acting on  $\alpha$  and  $d$  we make  $x_1 = 0; 1$ .

In result we obtain the following values  $(x, y, z, x', y', z')$ :

$$(1, 0, 1, 0, 0, 0),$$

$$(0, 0, 1, 0, 0, 0),$$

$$(1, 0, 0, 0, 0, 0),$$

$$(0, 0, 0, 0, 0, 0).$$

We require to note that for all the isolated values of the form  $(x, y, z, x', y', z')$ , the subalgebras so obtained are no more changing, under the action of the automorphism of Lie algebras.

Summarizing the conducted examination, one can formulate the theorem:

**Theorem III.7.1** Any Bol algebra of dimension 3, with the trilinear operation of Type VII and the canonical enveloping Lie algebra  $\mathfrak{G}$  of dimension 5, is isomorphic to one of Bol algebras below:

- |  |   |
|--|---|
| 1. Trivial                                   | $(e_2, e_3, e_1) = e_1$                                     |
| bilinear                                     | $(e_3, e_1, e_2) = e_1$                                     |
| operation                                    |   |
| 2. $e_2 \cdot e_3 = -e_1$                    | $(e_2, e_3, e_1) = e_1$                                     |
|  | $(e_3, e_1, e_2) = e_1$                                     |
| 3. $e_2 \cdot e_3 = -e_3$                    | $(e_2, e_3, e_1) = e_1$                                     |
|  | $(e_3, e_1, e_2) = e_1$                                     |
| 4. $e_2 \cdot e_3 = -e_1$                    | $(e_2, e_3, e_1) = e_1$                                     |
|  | $(e_3, e_1, e_2) = e_1$                                     |
| 5. $e_2 \cdot e_3 = e_1 - e_3$               | $(e_2, e_3, e_1) = e_1$                                     |
|  | $(e_3, e_1, e_2) = e_1$                                     |
| 6. $e_2 \cdot e_3 = -\omega e_1$             | $(e_2, e_3, e_1) = e_1, \omega > 0$                         |
| $e_1 \cdot e_3 = -e_1$                       | $(e_3, e_1, e_2) = e_1$                                     |
| 7. $e_2 \cdot e_3 = -se_1$                   | $(e_2, e_3, e_1) = e_1, s \geq 0$                           |
| $e_1 \cdot e_3 = -\lambda e_1 - e_2 - e_3$   | $(e_3, e_1, e_2) = e_1, \lambda > 0$                        |
| 8. $e_2 \cdot e_3 = -se_1$                   | $(e_2, e_3, e_1) = e_1, s \geq 0$                           |
| $e_1 \cdot e_3 = -\lambda e_1 - e_2$         | $(e_3, e_1, e_2) = e_1, \lambda > 0$                        |
| 9. $e_2 \cdot e_3 = -\gamma e_1 - e_3$       | $(e_2, e_3, e_1) = e_1, \gamma > 0$                         |
| $e_1 \cdot e_3 = -te_1 - e_3$                | $(e_3, e_1, e_2) = e_1, t > 0$                              |
| 10. $e_2 \cdot e_3 = -\eta e_1 - \kappa e_3$ | $(e_2, e_3, e_1) = e_1, \eta, \tau \geq 0$                  |
| $e_1 \cdot e_3 = -\tau e_1 - e_3,$           | $(e_3, e_1, e_2) = e_1 \kappa \geq 0$ but $\kappa \neq 1$ . |



In addition the distinguished Bol algebras are, not isomorphic among themselves.

We pass to the description of Bol 3-Webs corresponding to the isolated Bol algebras of Type VII. We note for the selected Bol algebras above, we will limit ourselves to the description of Bol 3-Webs from Bol algebra number 6.

The composition law  $(\triangle)$ , corresponding to the Lie group  $G$ , of Lie algebra enveloping Bol algebra is defined as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \triangle \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} x_1 + (1 + x_2 x_3) y_1 \exp(x_4 - \frac{x_2 x_3}{2}) \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 + \frac{x_2 y_3 + y_2 x_3}{2} \\ x_5 - x_3 y_1 \exp(x_4 - \frac{x_2 x_3}{2}) + y_5 \exp(x_4 - \frac{x_2 x_3}{2}) \end{bmatrix}.$$

The collection of elements in  $B$  is define as:

$$B = \exp \mathfrak{B} = \{t(2 \int_0^1 \exp(-\frac{uv\alpha^2}{2}) d\alpha - \exp \frac{-uv}{2}), u, v, \frac{t}{u}(\exp \frac{-uv}{2} - 1)\}_{t,u,v \in \mathbb{R}}$$

For a convenience of examination we will divide it into subcases:

1. 1. If  $u=0$  and  $v$ -any number,

$$B = \exp \mathfrak{B} = \{t, 0, v, \frac{-vt}{2}\}$$

2. 2. If  $v=0$  and  $u$ - any number,

$$B = \exp \mathfrak{B} = \{t, u, 0, 0\}.$$

3. 3. If  $u \neq 0, v \neq 0$  and have same sign then,

$$B = \exp \mathfrak{B} = \{t(2\beta\sqrt{\frac{2}{uv}} - \exp \frac{-uv}{2}), u, v, \frac{t}{u}(\exp \frac{-uv}{2} - 1)\}_{t,u,v \in \mathbb{R}},$$

$$\text{where } \beta = \int_0^1 \exp -p^2 dp \text{ and } p = \frac{uv}{2} \alpha^2.$$

Let Bol algebra be defined as follows:

$$e_2 \cdot e_3 = -\omega e_1 \quad (e_2, e_3, e_1) = e_1, \omega > 0$$

$$e_1 \cdot e_3 = -e_1 \quad (e_3, e_1, e_2) = e_1$$

the subalgebra  $H = \exp \mathfrak{h}$  can be defined as:

$$H = \exp \mathfrak{h} = \{\exp \alpha(e_4 + \omega e_1), \exp l(e_5 + e_1)\}_{\alpha \in \mathbb{R}} = \{l + \alpha\omega, 0, 0, \alpha, l\}_{\alpha, l \in \mathbb{R}}.$$

The collection of elements  $B = \exp \mathfrak{B}$  is:

1. 1. If  $u=0$  and  $v$ -any number'

$$B = \exp \mathfrak{B} = \{t, 0, v, \frac{-vt}{2}\}.$$

Hence, any element  $(x_1, x_2, x_3, x_4, x_5)$  from  $G$  can not be always represented as an element of  $B$  and  $H$ .

2. 2. If  $v=0$  and  $u$ - any number,

$$B = \exp \mathfrak{B} = \{t, u, 0, 0\}.$$

In this case too, any element from  $G$ , can not be always represented as an element of  $B$  and  $H$ .

3. 3. If  $u \neq 0, v \neq 0$  and have same sign then,

$$B = \exp \mathfrak{B} = \{t(2\beta\sqrt{\frac{2}{uv}} - \exp \frac{-uv}{2}), u, v, \frac{t}{u}(\exp \frac{-uv}{2} - 1)\}_{t,u,v \in \mathbb{R}}$$

where  $\beta = \int_0^1 \exp -p^2 dp$  and  $p = \frac{uv}{2}\alpha^2$

Here, any element  $(x_1, x_2, x_3, x_4, x_5)$  from  $G$  sufficiently in the neighborhood of  $e$  can be uniquely represented as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{bmatrix} P(2\beta\sqrt{\frac{2}{x_2x_3}} - e^{-\frac{x_2x_3}{2}}) \\ x_2 \\ x_3 \\ 0 \\ P\frac{e^{-\frac{x_2x_3}{2}} - 1}{x_2} \end{bmatrix} \Delta \begin{bmatrix} L + \omega x_4 \\ 0 \\ 0 \\ x_4 \\ L \end{bmatrix}$$

where  $P, L$  are defined from the relations

$$P = \frac{We^{-\frac{x_2x_3}{2}}}{S},$$

$$L = \frac{Q}{Z},$$

$$K = \frac{e^{-\frac{x_2x_3}{2}} - 1}{x_2},$$

$$W = [(x_1 - \omega x_4(1 + x_2x_3))(1 - x_3) - (1 - x_2 + x_2x_3)(x_5 + \omega x_3x_4)]$$

$$S = (2\beta\sqrt{\frac{2}{x_2x_3}} - e^{-\frac{x_2x_3}{2}})(1 - x_3)e^{-\frac{x_2x_3}{2}} - (1 - x_2 + x_2x_3)e^{-\frac{x_2x_3}{2}}K,$$

$$Q = (2\beta\sqrt{\frac{2}{x_2x_3}} - e^{-\frac{x_2x_3}{2}})(x_5 + \omega x_3x_4e^{-\frac{x_2x_3}{2}}) - (x_1 - \omega x_4(1 + x_2x_3)e^{-\frac{x_2x_3}{2}})K,$$

$$Z = (2\beta\sqrt{\frac{2}{x_2x_3}} - e^{-\frac{x_2x_3}{2}})(1 - x_3)e^{-\frac{x_2x_3}{2}} - (1 - x_2 + x_2x_3)e^{-\frac{x_2x_3}{2}}K.$$

The composition law  $(\star)$ , corresponding to the local analytical Bol loop  $B(\star)$  is defined as follows:

$$\begin{aligned} &= \exp^{-1} \left( \prod_B \begin{bmatrix} t + t'(1 + uv)e^{-\frac{uv}{2}} \\ u + u' \\ v + v' \\ \frac{uv' - vu'}{2} \\ vt'e^{-\frac{uv}{2}} \end{bmatrix} \right) \\ &= \exp^{-1} \left( \begin{bmatrix} P_1(2\beta\sqrt{\frac{2}{(u+u')(v+v')}} - e^{-\frac{(u+u')(v+v')}{2}}) \\ u + u' \\ v + v' \\ 0 \\ T_1 \frac{e^{-\frac{(u+u')(v+v')}{2}} - 1}{u+u'} \end{bmatrix} \right) \\ &= \begin{bmatrix} P_1 \\ u + u' \\ v + v' \end{bmatrix}. \end{aligned}$$

Where:

$$T_2 = \left\{ t + t'(1 + uv)e^{-\frac{uv}{2}} - \omega \frac{uv' - vu'}{2} [1 + (u + u')(v + v')] \right\} [1 - (u + u')],$$

$$R_2 = [1 - u - u' + (u + u')(v + v')][vt'e^{-\frac{uv}{2}} + \omega(v + v')\frac{(uv' - vu')}{2}]$$

$$M_2 = \left( 2\beta\sqrt{\frac{2}{(u + u')(v + v')}} - e^{-\frac{(u+u')(v+v')}{2}} \right) [1 - (v + v')]e^{-\frac{(u+u')(v+v')}{2}},$$

$$N_2 = [1 - u - u' + (u + u')(v + v')]e^{-\frac{(u+u')(v+v')}{2}} \frac{(e^{-\frac{(u+u')(v+v')}{2}} - 1)}{u + u'}$$

$$P_1 = \frac{T_2 - R_2}{M_2 - N_2}.$$

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